Department of Mathematics

Faculty of Engineering &Technology

V.B.S. Purvanchal University, Jaunpur

Prepared by: Dr. Sushil Shukla

Study Material

Syllabus: Module 3: Fourier Series and Laplace Transforms: Dirichlet's conditions, Fourier series including half range and its applications to find the sum of infinite series; Laplace Transforms: Definition, Necessary and sufficient conditions for the existence of Laplace transform, initial and final value theorems, Laplace transform of derivatives and integrals, Unit step function, Dirac-delta function, Laplace transform of Periodic function, Inverse Laplace Transform, Convolution theorem (without proof) and application to solve linear and nonlinear ODE.

Dirichlet's Conditions: For $f(x)$ to be expandable in a Fourier series over $(-\pi, \pi)$, it must satisfy:

1. $f(x)$ is single-valued and bounded.
2. It has a finite number of maxima, minima, and discontinuities.
3. $f(x + 2\pi) = f(x)$ outside $[-\pi, \pi]$.

At continuous points, the series converges to $f(x)$. At points of discontinuity, it converges to:

$$\frac{1}{2}[f(x + 0) + f(x - 0)]$$

Advantages of Fourier Series: Discontinuous functions can be represented (unlike Taylor series).

- Useful for expanding functions periodic only outside a closed interval.
- Expands oscillating functions into all oscillation modes.
- Term-by-term integration is generally valid.

Useful Integrals:

(i) $\int_0^{2\pi} \text{Sin}\, nxdx = 0$ (ii) $\int_0^{2\pi} \text{Cos}\, nxdx = 0$ (iii) $\int_0^{2\pi} \text{Sin}^2\, nxdx = \pi$ (iv)

$\int_0^{2\pi} \text{Cos}^2\, nxdx = \pi$ (v) $\int_0^{2\pi} \text{Sin}\, nx \text{Sin}\, mxdx = 0$

(vi) $\int_0^{2\pi} \text{Cos}\, nx.\text{Cos}\, mxdx = 0$ (vii) $\int_0^{2\pi} \text{Sin}\, nx.\text{Cos}\, mxdx = 0$

(viii) $\int_0^{2\pi} \text{Sin}\, nx.\text{Cos}\, nxdx = 0$ (ix) $\text{Sin}\, n\pi = 0$ (x) $\text{Cos}\, n\pi = (-1)^n$ *where* $n \in I$

(xi) $\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...$

where $u' = \frac{du}{dx}, u'' = \frac{d^2u}{dx^2} + ...$

$v_1 = \int vdx, \; v_2 = \int v_1 dx$

Periodic Functions: A function f(t) is periodic if its value repeats at equal intervals on the horizontal axis:

$$f(t) = f(t + T) = f(t + 2T) = \cdots$$

Here, T is the period of f(t). For example, $\sin x = \sin(x + 2\pi) = \sin(x + 4\pi)$, so $\sin x$ is periodic with period 2π.

Fourier Series: A non-sinusoidal periodic function can be expanded into a series of sines and cosines:

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$$

In summation notation:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

- a_0: Constant term (d.c. component in electrical systems)
- a_1, b_1: Fundamental frequency components
- a_n, b_n: Harmonics and Fourier coefficients

Determination of Fourier Coefficients (Euler's Formula) :

$$f(x) = \frac{a_0}{2} + a_1 \operatorname{Cos} x + a_2 \operatorname{Cos} 2x + a_3 \operatorname{Cos} 3x + \ldots\ldots\ldots\ldots + a_n \operatorname{Cos} nx + \ldots\ldots$$
$$+ b_1 \operatorname{Sin} x + b_2 \operatorname{Sin} 2x + \ldots\ldots\ldots\ldots + b_n \operatorname{Sin} nx + \ldots\ldots\ldots(1)$$

(1) To find a_0 : Integrate both side of (1) from x = 0 to x = 2π

$$\int_0^{2\pi} f(x)dx = \frac{a_0}{2}\int_0^{2\pi} dx + a_1\int_0^{2\pi} \operatorname{Cos} x dx + a_2\int_0^{2\pi} \operatorname{Cos} 2x dx + \ldots\ldots\ldots\ldots$$
$$+ a_n\int_0^{2\pi} \operatorname{Cos} nx dx + b_1\int_0^{2\pi} \operatorname{Sin} x dx + b_2\int_0^{2\pi} \operatorname{Sin} 2x dx + \ldots\ldots\ldots\ldots$$
$$+ b_n\int_0^{2\pi} \operatorname{Sin} nx dx + \ldots\ldots\ldots$$

$$= \frac{a_0}{2}\int_0^{2\pi} dx \qquad \text{other integrals 0 by formula (1) and (2)}$$

$$= \frac{a_0}{2} \times \pi$$
$$= a_0\pi$$

so $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

(2) To find a_n : Multiply each side of (1) by Cos nx and integrate from x =0 to 2π

$$\int_0^{2\pi} f(x) \operatorname{Cos} nx dx = \frac{a_0}{2}\int_0^{2\pi} \operatorname{Cos} nx dx + a_1\int_0^{2\pi} \operatorname{Cos} x \operatorname{Cos} nx dx + \ldots\ldots\ldots\ldots$$
$$+ a_n\int_0^{2\pi} \operatorname{Cos}^2 nx dx + b_1\int_0^{2\pi} \operatorname{Sin} x \operatorname{Cos} nx dx + b_2\int_0^{2\pi} \operatorname{Sin} 2x \operatorname{Cos} nx dx + \ldots\ldots\ldots\ldots$$

$$= a_n\int_0^{2\pi} \operatorname{Cos}^2 nx dx = a_n\pi \qquad \text{other integrals = 0 by sd. formula}$$

so $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \operatorname{Cos} nx dx$

(3) To find b_n : Multiply each side of (1) by Cos nx and integrate from x =0 to 2π

$$\int_0^{2\pi} f(x) \operatorname{Sin} nx dx = \frac{a_0}{2}\int_0^{2\pi} \operatorname{Sin} nx dx + a_1\int_0^{2\pi} \operatorname{Cos} x \operatorname{Sin} nx dx + \ldots\ldots\ldots\ldots$$
$$+ a_n\int_0^{2\pi} \operatorname{Cos} nx \operatorname{Sin} nx dx + b_1\int_0^{2\pi} \operatorname{Sin} x \operatorname{Sin} nx dx + b_2\int_0^{2\pi} \operatorname{Sin} 2x \operatorname{Sin} nx dx + \ldots\ldots\ldots\ldots$$

$$= b_n\int_0^{2\pi} \operatorname{Sin}^2 nx dx = b_n\int_0^{2\pi} \operatorname{Sin}^2 nx dx = b_n\pi \qquad \text{by sd. formula}$$

so $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\text{Sin}\,nx\,dx$

Problem: Find the Fourier Series representation of *f(x) = x, 0 < x < 2π*

Solution : Let $f(x) = \frac{a_0}{2} + a_1\,\text{Cos}\,x + a_2\,\text{Cos}\,2x + \ldots\ldots\ldots + b_1\,\text{Sin}\,x + b_2\,\text{Sin}\,2x + \ldots\ldots\ldots$

Hence $= a_0 \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\text{Cos}\,nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,\text{Cos}\,nx\,dx$$

$$= \frac{1}{\pi}\left[x.\frac{\text{Sin}\,x}{n} - 1.\left(\frac{-\text{Cos}\,nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{\text{Cos}\,2n\pi}{n} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\text{Sin}\,nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,\text{Sin}\,nx\,dx$$

$$= \frac{1}{\pi}\left[x.\left(-\frac{\text{Cos}\,x}{n}\right) - 1.\left(\frac{-\text{Sin}\,nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-2\pi\,\text{Cos}\,2n\pi}{n}\right] = -\frac{2}{n}$$

Substituting the values of $a_0, a_1, a_2 \ldots\ldots\ldots b_1, b_2 \ldots\ldots\ldots$ in (1) we have

$$x = \pi - 2\left[\text{Sin}\,x + \frac{1}{2}\text{Sin}\,2x + \frac{1}{3}\text{Sin}\,3x + \ldots\ldots\right]$$

Problem : Given that $f(x) = x + x^2$ *for* $-\pi < x < \pi$, find the Fourier expression of f(x).

Deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots$

Solution : Let $x + x^2 = \frac{a_0}{2} + a_1\,\text{Cos}\,x - a_2\,\text{Cos}\,2x + \ldots\ldots + b_1\,\text{Sin}\,x + b_2\,\text{Sin}\,2x + \ldots\ldots$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)dx = \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{Cos}\,nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\,\text{Cos}\,nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\text{Sin}\,nx}{n} - (2x+1)\left(\frac{-\text{Cos}\,nx}{n^2}\right) + (2)\left(-\frac{\text{Sin}\,nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{\text{Cos}\,nx}{n^2} - (2\pi+1)\frac{\text{Cos}(-n\pi)}{n^2}\right] = \frac{1}{\pi}\left[4\pi\frac{\text{Cos}\,n\pi}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{Sin}\,nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,\text{Sin}\,nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\text{Cos}\,nx}{n}\right) - (2x+1)\left(\frac{-\text{Sin}\,nx}{n^2}\right) + 2\frac{\text{Cos}\,nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\text{Cos}\,n\pi}{n} + 2\frac{\text{Cos}\,n\pi}{n^3} + (-\pi+\pi^2)\frac{\text{Cos}\,n\pi}{n} - 2\frac{\text{Cos}\,n\pi}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\text{Cos}\,n\pi\right] = -\frac{2}{n}(-1)^n$$

Substituting the values of a_0, a_n, b_n in (1) we get:

$$x + x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$
$$- 2\left[-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right] \quad \ldots (2)$$

Put $x = \pi$ in (2), $\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$ …… (3)

Put $x = -\pi$ in (2), $-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$ ……. (4)

Adding (3) and (4):

$$2\pi^2 = \frac{2\pi^2}{3} + 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\frac{4\pi^2}{3} = 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Problem: Find the Fourier series for the periodic function:

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$
$$f(x + 2\pi) = f(x)$$

Solution. Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \cdots + b_1\sin x + b_2\sin 2x + \cdots$ (1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 0 \cdot dx + \frac{1}{\pi}\int_{0}^{\pi} x\, dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} x\cos n\,x\,dx = \frac{1}{\pi}\left[x \cdot \frac{\sin n\,x}{n} - (1)\left(-\frac{\cos n\,x}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = -\frac{2}{n^2\pi} \quad \text{when n is odd} \quad = 0 \quad \text{when n is even.}$$

$$b_n = \frac{1}{\pi}\int_{0}^{\pi} x\sin n\,x\,dx = \frac{1}{\pi}\left[x\left(-\frac{\cos n\,x}{n}\right) - (1)\left(-\frac{\sin n\,x}{n^2}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right]$$

$$= -\frac{(-1)^n}{n}$$

Substituting the values of $a_0, a_1, a_2 \ldots b_1, b_2, \ldots$ in (1), we get:

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3\,x}{3^2} + \frac{\cos 5\,x}{5^2} \ldots\right] + \left[\frac{\sin x}{1} - \frac{\sin 2\,x}{2} + \frac{\sin 3\,x}{3} + \cdots\right]$$

Discontinuous Functions:

At a point of discontinuity, Fourier series gives the value of f(x) as the arithmetic mean of left and right limits.

At the point of discontinuity, $x = c$

$$\text{At } x = c, f(x) = \frac{1}{2}[f(c-0) + f(c+0)]$$

Problem: Find the Fourier series for f(x), if $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Solution. Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2\,x + \cdots + a_n\cos n\,x + \cdots + b_1\sin x + b_2\sin 2\,x + \cdots + b_n\sin n\,x + \cdots$ ... (1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$\text{Then } a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_{0}^{\pi} x\,dx\right] = \frac{1}{\pi}[-\pi(x)_{-\pi}^{0} + (x^2/2)_0^{\pi}] = \frac{1}{\pi}(-\pi^2 + \pi^2/2)$$

$$= -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos n\,x\,dx$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos n\,x\,dx + \int_{0}^{\pi} x\cos n\,x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin n\,x}{n}\right)_{-\pi}^{0} + \left(\frac{x\sin n\,x}{n} + \frac{\cos n\,x}{n^2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\,\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\,\pi - 1)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin n\,x\,dx$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin n\,x\,dx + \int_{0}^{\pi} x\sin n\,x\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi\cos n\,x}{n}\right)_{-\pi}^{0} + \left(-x\frac{\cos n\,x}{n} + \frac{\sin n\,x}{n^2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\,\pi) - \frac{\pi}{n}\cos n\,\pi\right] = \frac{1}{n}(1 - 2\cos n\,\pi)$$

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} \quad \ldots(2)$$

Putting $x = 0$ in (2), we get $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\infty\right)$ ... (3)

Now $f(x)$ is discontinuous at $x = 0$.

But $f(0-0) = -\pi$ and $f(0+0) = 0$ $\quad \therefore f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\pi/2$

From (3), $-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$

$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Problem: Find the Fourier series expansion of the periodic function of period 2π, defined by

$$f(x) = x, \quad \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad f(x) = \pi - x, \quad \text{if } \frac{\pi}{2} < x < \frac{3\pi}{2}$$

Solution. Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \cdots + b_1\sin x + b_2\sin 2x + \cdots$

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2}(\pi - x)\,dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{3\pi/2}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{8} - \frac{\pi^2}{8}\right) + \frac{1}{\pi}\left(\frac{3\pi^2}{2} - \frac{9\pi^2}{8} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right) = \pi\left(\frac{3}{2} - \frac{9}{8} - \frac{1}{2} + \frac{1}{8}\right) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\cos n\,x\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} (\pi - x)\cos n\,x\,dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin n\,x}{n} - (1)\left(-\frac{\cos n\,x}{n^2}\right)\right]_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left[(\pi - x)\frac{\sin n\,x}{n} - (-1)\left(-\frac{\cos n\,x}{n^2}\right)\right]_{\pi/2}^{3\pi/2}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} - \frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$+ \frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\sin\frac{3n\pi}{2}}{n} - \frac{\cos\frac{3n\pi}{2}}{n^2} - \frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\left(\sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\,\pi\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin\frac{n\pi}{2}\sin n\,\pi\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin n\,x\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} (\pi - x)\sin n\,x\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2} x\sin n\,x\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} (\pi - x)\sin n\,x\,dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos n\,x}{n}\right) - (1)\left(-\frac{\sin n\,x}{n^2}\right)\right]_{0}^{\pi/2}$$

$$+ \frac{1}{\pi}\left[(\pi - x)\left(-\frac{\cos n\,x}{n}\right) - (-1)\left(-\frac{\sin n\,x}{n^2}\right)\right]_{\pi/2}^{3\pi/2}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \frac{1}{\pi}\left[\frac{\pi}{2}\frac{\cos\frac{3n\pi}{2}}{n} - \frac{\sin\frac{3n\pi}{2}}{n^2} + \frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{3\sin\frac{n\pi}{2}}{n^2} + \frac{\pi}{2}\frac{\cos\frac{3n\pi}{2}}{n} - \frac{\sin\frac{3n\pi}{2}}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right) + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin\frac{n\pi}{2}\sin n\,\pi + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right] = \frac{1}{n^2\pi}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right]$$

Substituting the values of $a_0, a_1, a_2 \ldots b_1, b_2, \ldots$ we get:

$$f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3\,x}{3^2} + \frac{\sin 5\,x}{5^2} - \cdots\right]$$

Problem: Find the Fourier series of the function defined as:

$$f(x) = \begin{cases} x + \pi & \text{for } 0 \le x \le \pi \\ -x - \pi & \text{for } -\pi \le x < 0 \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x).$$

Solution. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\,dx + \frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\,dx = \frac{1}{\pi}\left(-\frac{x^2}{2}-\pi x\right)_{-\pi}^{0} + \frac{1}{\pi}\left(\frac{x^2}{2}+\pi x\right)_{0}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2}-\pi^2\right) + \frac{1}{\pi}\left(\frac{\pi^2}{2}+\pi^2\right) = \pi\left(\frac{1}{2}-1\right) + \pi\left(\frac{1}{2}+1\right) = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos n x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos n x\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos n x\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\cos n x\,dx + \frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\cos n x\,dx$$

$$= \frac{1}{\pi}\left[(-x-\pi)\frac{\sin n x}{n} - (-1)\left\{-\frac{\cos n x}{n^2}\right\}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(x+\pi)\frac{\sin n x}{n} - (1)\left\{-\frac{\cos n x}{n^2}\right\}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2} + \frac{(-1)^n}{n^2}\right] + \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}[(-1)^n - 1] = \frac{-4}{n^2\pi} \quad \text{if n is odd.}$$

$$= 0 \quad \text{if n is even.}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin n x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin n x\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin n x\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\sin n x\,dx + \frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\sin n x\,dx$$

$$= \frac{1}{\pi}\left[(-x-\pi)\left(-\frac{\cos n x}{n}\right) - (-1)\left(-\frac{\sin n x}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi}\left[(x+\pi)\left(-\frac{\cos n x}{n}\right) - (1)\left(-\frac{\sin n x}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\right] + \frac{1}{\pi}\left[-\frac{2\pi}{n}(-1)^n + \frac{\pi}{n}\right] = \frac{1}{n}[(1) - 2(-1)^n + (1)] = \frac{2}{n}[1-(-1)^n]$$

$$= \frac{4}{n}, \quad \text{if n is odd.}$$

$$= 0, \quad \text{if n is even.}$$

Fourier series is:

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \cdots + b_1\sin x + b_2\sin 2x + \cdots$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right) + 4\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots\right)$$

Problem: Find the Fourier Series of the function

$$f(x) = \begin{cases} -1 & for \quad -\pi < x < -\dfrac{\pi}{2} \\ 0 & for \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ +1 & for \quad \dfrac{\pi}{2} < x < \pi \end{cases}$$

Solution : Let $f(x) = \frac{a_0}{2} + a_1 \text{Cos}\, x + a_2 \text{Cos}\, 2x + ... + b_1 \text{Sin}\, x + b_2 \text{Sin}\, 2x + ...$ .....(1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(-1)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} 0.dx + \frac{1}{\pi}\int_{-\pi}^{\pi} 1.dx$$

$$= \frac{1}{\pi}\left[-x\right]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left[x\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi}\left[-x\right]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left[x\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \pi - \frac{\pi}{2}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{Cos}\, nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi}(-1)\,\text{Cos}\, nxdx + \frac{1}{\pi}\int_{-\pi}^{\pi}(0)\,\text{Cos}\, nxdx + \frac{1}{\pi}\int_{-\pi}^{\pi}(1)\,\text{Cos}\, nxdx$$

$$= -\frac{1}{\pi}\left[\frac{\text{Sin}\, nx}{n}\right]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left[\frac{\text{Sin}\, nx}{n}\right]_{\pi/2}^{\pi} = -\frac{1}{\pi}\left[-\frac{\text{Sin}\,\frac{n\pi}{2}}{n}\right] + \frac{1}{\pi}\left[-\frac{\text{Sin}\,\frac{n\pi}{2}}{n}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{Sin}\, nxdx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\,\text{Sin}\, nxdx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\,\text{Sin}\, nxdx + \frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\,\text{Sin}\, nxdx$$

$$= \frac{1}{\pi}\left[\frac{\text{Cos}\, nx}{n}\right]_{-\pi}^{-\pi/2} - \frac{1}{\pi}\left[\frac{\text{Cos}\, nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\left[\text{Cos}\,\frac{n\pi}{2} - \text{Cos}\, n\pi\right] - \frac{1}{n\pi}\left(\text{Cos}\, n\pi - \text{Cos}\,\frac{n\pi}{2}\right) = \frac{2}{n\pi}\left[\text{Cos}\,\frac{n\pi}{2} - \text{Cos}\, n\pi\right]$$

Even Function & Odd function:

A function is said to be even (or symmetric) function, if f(-x) = f(x). Graph of such functions is symmetrical with respect to y-axis. Here y - axis is a mirror for the reflection of the curve.

The area under such a curve from -π to π is double the area from 0 to π.

$$\int_{-\pi}^{\pi} f(x)\, dx = 2\int_{0}^{\pi} f(x)\, dx$$

Odd function: A function f(x) is called odd (or skew symmetric) function if

$$f(-x) = -f(x)$$

here the area under the curve -π to π is zero.

Representation of even function as Fourier Series:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{Cos}\;nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,\text{Cos}\;nx\,dx$$

as *f(x)* and Cos nx both are even function. So product of *f(x).* Cos nx is also an even function. $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\;\text{Sin}\,nx\,dx = 0$

As Sin nx is an odd function so f(x). Sin nx is also an odd function, we need not to calculate bn as its value = 0. So series of even functions contain only cos terms.

Representation of an odd function as Fourier series.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\;\text{Cos}\;nx\,dx = 0 \quad [\,f(x)\,\text{Cos}\,nx \;\; is\ on\ odd\ function\,]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{Sin}\;nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,\text{Sin}\,nx\,dx$$

$$[\,f(x)\,\text{Sin}\,nx \;\; is\ on\ even\ function\,]$$

So the series of odd functions will contain only sine terms.

Problem: Find the Fourier series expansion of the periodic function of period 2π

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

Hence, find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Solution.

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

This is an even function. $\therefore b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos n\,x\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos n\,x\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin n\,x}{n}\right) - (2x)\left(-\frac{\cos n\,x}{n^2}\right) + (2)\left(-\frac{\sin n\,x}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2 \sin n\,\pi}{n} + \frac{2\pi\cos n\,\pi}{n^2} - \frac{2\sin n\,\pi}{n^3}\right] = \frac{4(-1)^n}{n^2}$$

Fourier series is $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + a_n \cos nx + \cdots$

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right]$$

On putting $x = 0$, we have

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots\right]$$

$$\text{or } \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots = \frac{\pi^2}{12}$$

Problem: Obtain a Fourier expression for

$$f(x) = x^3 \quad \text{for } -\pi < x < \pi$$

Solution. $f(x) = x^3$ is an odd function.

$$\therefore a_0 = 0 \quad \text{and } a_n = 0$$

$$\cdot b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\, dx$$

$$\left[\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\right]$$

$$= \frac{2}{\pi}\left[x^3\left(-\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3 \cos n\pi}{n} + \frac{6\pi \cos n\pi}{n^3}\right] = 2 \cdot (-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

$$\therefore x^3 = 2\left[-\left(-\frac{\pi^2}{1} + \frac{6}{1^3}\right)\sin x + \left(-\frac{\pi^2}{2} + \frac{6}{2^3}\right)\sin 2x - \left(-\frac{\pi^2}{3} + \frac{6}{3^3}\right)\sin 3x \ldots\right]$$

Half-Range Series, period 0 to π:

If given function is defined in the interval (0, π) and it is immaterial whatever the function may be outside the internal (0, π).

To get the series of cosines only we assume that f(x) is an even function in the interval (-π, π). $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \text{ Cos } nx\, dx \text{ and } b_n = 0$

To expand f(x) as a sine series we extend the function in the interval (-π, π) as an odd function $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \text{ Sin } nx\, dx \text{ and } a_n = 0$

Problem: Represent the following function by a Fourier sin series:

$$f(t) = \begin{cases} t, & 0 < t \le \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} < t \le \pi \end{cases}$$

Solution. $b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin n\,t\,dt$

$$= \frac{2}{\pi}\int_0^{\pi/2} t\sin n\,t\,dt + \frac{2}{\pi}\int_{\pi/2}^{\pi}\frac{\pi}{2}\sin n\,t\,dt$$

$$= \frac{2}{\pi}\left[t\left(-\frac{\cos n\,t}{n}\right) - (1)\left(-\frac{\sin n\,t}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\frac{\pi}{2}\left[-\frac{\cos n\,t}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \left[-\frac{\cos n\,\pi}{n} + \frac{\cos\frac{n\pi}{2}}{n}\right]$$

$$b_1 = \frac{2}{\pi}\left[-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right] + \left[-\cos\pi + \cos\frac{\pi}{2}\right] = \frac{2}{\pi}[0+1] + [1] = \frac{2}{\pi} + 1$$

$$b_2 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\pi}{2} + \frac{\sin\pi}{2^2}\right] + \left[-\frac{\cos 2\,\pi}{2} + \frac{\cos\pi}{2}\right] = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{(-1)}{2} + 0\right] + \left[-\frac{1}{2} - \frac{1}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\right] - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$b_3 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{3\pi}{2}}{3} + \frac{\sin\frac{3\pi}{2}}{3^2}\right] + \left[-\frac{\cos 3\,\pi}{3} + \frac{\cos\frac{3\pi}{2}}{3}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}(0) - \frac{1}{9}\right] + \left[\frac{1}{3} + 0\right] = -\frac{2}{9\pi} + \frac{1}{3}$$

$$f(t) = \left(\frac{2}{\pi} + 1\right)\sin t - \frac{1}{2}\sin 2\,t + \left(-\frac{2}{9\pi} + \frac{1}{3}\right)\sin 3\,t + \cdots$$

Problem: Find the Fourier sine series for the function $f(x) = e^{ax}$ for $0 < x < \pi$ where a is constant.

Solution. $\int e^{ax}\sin b\,x\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin b\,x - b\cos b\,x]$

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{ax}\sin n\,x\,dx$$

$$= \frac{2}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin n\,x - n\cos n\,x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(a\sin n\,\pi - n\cos n\,\pi) + \frac{n}{a^2+n^2}\right]$$

$$= \frac{2}{\pi}\frac{n}{a^2+n^2}[-(-1)^n e^{a\pi} + 1] = \frac{2n}{(a^2+n^2)\pi}[1 - (-1)^n e^{a\pi}]$$

$$b_1 = \frac{2(1+e^{a\pi})}{(a^2+1^2)\pi}, \quad b_2 = \frac{2\cdot 2\cdot(1-e^{a\pi})}{(a^2+2^2)\pi}$$

$$e^{ax} = \frac{2}{\pi}\left[\frac{1+e^{a\pi}}{a^2+1^2}\sin x + \frac{2(2-e^{a\pi})}{a^2+2^2}\sin 2x + \cdots\right]$$

Change of Interval: (Functions having arbitrary period):

The period of function is not required be always 2π, T or 2C, in that case period must be converted to the length 2π & independent variable x changes proportionally.

Let the function be defined in the interval (-C, C). Now we have to function of period 2π so that we can use formula of a_n, b_n

2C is the interval of the variable x.

1 is the interval of the variable $\frac{x}{2C}$

2π is the interval of the variable $\frac{x\times 2\pi}{2C} = \frac{\pi x}{C} = Z(Let)$ *or* $x = \frac{Zc}{\pi}$

$$\text{so put } z = \frac{\pi x}{c} \quad \text{or } x = \frac{zc}{\pi}$$

Thus the function f(x)of period 2cis transformed to the function $f\left(\frac{cz}{\pi}\right)$or F(z)of period 2π.

F(z)can be expanded in the Fourier series.

$$F(z) = f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + a_1\cos z + a_2\cos 2z + b_1\sin z + b_2\sin 2z + \cdots$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} F(z)\,dz = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cz}{\pi}\right)\,dz$

$$= \frac{1}{c}\int_0^{2c} f(x)\,d\left(\frac{\pi x}{c}\right) = \frac{1}{c}\int_0^{2c} f(x)\,dx \quad \text{put } z = \frac{\pi x}{c}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} F(z)\cos nz\,dz = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cz}{\pi}\right)\cos nx\,dz \quad \text{Put } z = \frac{\pi x}{c}$$

$$= \frac{1}{\pi}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\,d\left(\frac{\pi x}{c}\right) = \frac{1}{c}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\,dx.$$

$$\text{Similarly } b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{c}\,dx.$$

Cor. Half range series [Interval $(0, c)$] Cosine series:

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + \cdots + a_n\cos\frac{n\pi x}{c} + \cdots$$

$$\text{where } a_0 = \frac{2}{c}\int_0^{c} f(x)\,dx, \quad a_n = \frac{2}{c}\int_0^{c} f(x)\cos\frac{n\pi x}{c}\,dx$$

Sine series.

$$f(x) = b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + \cdots + b_n\sin\frac{n\pi x}{c} + \cdots$$

$$\text{where } b_n = \frac{2}{c}\int_0^{c} f(x)\sin\frac{n\pi x}{c}\,dx.$$

Problem: A periodic function of period 4 is defined as

$$f(x) = |x|, \quad -2 < x < 2.$$

Find its Fourier series expansion.

Solution. $f(x) = |x| \quad -2 < x < 2$

$$f(x) = x \quad 0 < x < 2$$

$$= -x \quad -2 < x < 0$$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2}\int_0^2 x\,dx + \frac{1}{2}\int_{-2}^{0}(-x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 + \frac{1}{2}\left[\frac{-x^2}{2}\right]_{-2}^{0} = \frac{1}{4}(4-0) + \frac{1}{4}(0+4) = 2$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx = \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$$

$$+\frac{1}{2}\left[(-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (-1)\left(-\frac{4}{n^2\pi^2}\right)\cos\frac{n\pi x}{2}\right]_{-2}^{0}$$

$$= \frac{1}{2}\left[0 + \frac{4}{n^2\pi^2}(-1)^n - \frac{4}{n^2\pi^2}\right] + \frac{1}{2}\left[0 - \frac{4}{n^2\pi^2} + \frac{4}{n^2\pi^2}(-1)^n\right]$$

$$= \frac{1}{2}\frac{4}{n^2\pi^2}[(-1)^n - 1 - 1 + (-1)^n] = \frac{4}{n^2\pi^2}[(-1)^n - 1]$$

$$= -\frac{8}{n^2\pi^2} \quad \text{if n is odd.}$$

$$= 0 \quad \text{if n is even.}$$

$b_n = 0$as $f(x)$is even function.

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + \cdots + b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + \cdots$$

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \frac{\cos\frac{5\pi x}{2}}{5^2} + \cdots\right]$$

Problem: Find Fourier half-range even expansion of the function,

$$f(x) = (-x/l) + 1, \quad 0 \le x \le l.$$

Solution. $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l\left(-\frac{x}{l} + 1\right)dx$

$$= \frac{2}{l}\left[-\frac{x^2}{2l} + x\right]_0^l = \frac{2}{l}\left[-\frac{l^2}{2l} + l\right] = \frac{2l}{l}\left[-\frac{1}{2} + 1\right] = 1$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l \left(-\frac{x}{l} + 1\right)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(-\frac{x}{l} + 1\right)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - \left(-\frac{1}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[0 - \frac{l}{n^2\pi^2}\cos n\,\pi + \frac{l}{n^2\pi^2}\right] = \frac{2}{l}\frac{l}{n^2\pi^2}[-(-1)^n + 1] = \frac{2}{n^2\pi^2}[1 - (-1)^n]$$

$$= \frac{4}{n^2\pi^2} \quad \text{when n is odd.}$$

$$= 0 \quad \text{when n is even.}$$

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} \ldots\right]$$

Problem: Expand $f(x) = e^x$ in a cosine series over $(0,1)$.

Solution. $f(x) = e^x$ and $c = 1$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{1}\int_0^1 e^x\,dx = 2(e - 1)$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx = \frac{2}{1}\int_0^1 e^x\cos\frac{n\pi x}{1}\,dx$$

$$= 2\left[\frac{e^x}{n^2\pi^2 + 1}(n\pi\sin n\,\pi x + \cos n\,\pi x)\right]_0^1$$

$$= 2\left[\frac{e^x}{n^2\pi^2 + 1}(n\pi\sin n\,\pi + \cos n\,\pi) - \frac{1}{n^2\pi^2 + 1}\right]$$

$$= \frac{2}{n^2\pi^2 + 1}[(-1)^n e - 1]$$

$$f(x) = \frac{a_0}{2} + a_1\cos\pi x + a_2\cos 2\,\pi x + a_3\cos 3\,\pi x + \cdots \ldots \ldots \ldots$$

$$e^x = e - 1 + 2\left[\frac{-e - 1}{\pi^2 + 1}\cos\pi x + \frac{e - 1}{4\pi^2 + 1}\cos 2\,\pi x + \frac{-e - 1}{9\pi^2 + 1}\cos 3\,\pi x + \cdots\right]$$

Problem: Obtain the half range sine series for the function $f(x) = x^2$ in the interval $0 < x < 3$.

Solution : Half range since series is given by $f(x) = \sum b_n \operatorname{Sin} nx$

where $b_n = \frac{2}{c}\int_0^c f(x)\ \operatorname{Sin}\frac{n\pi x}{C}\,dx$ in the half range (0, c).

Here we have half range $0 < x < 3$ & $f(x) = x^2$

$$b_n = \frac{2}{3}\int_0^c x^2 \ \text{Sin}\ \frac{n\pi x}{3}\, dx$$

$$= \frac{2}{3}\left[x^2.\frac{-3}{n\pi}\left(\text{Cos}\frac{n\pi x}{3}\right) + 2x\times\left(\frac{3}{n\pi}\right)\left(\frac{3}{n\pi}\right)\text{Sin}\frac{n\pi x}{3} - 2\left(\frac{3}{n\pi}\right)\left(\frac{3}{n\pi}\right)\left(\frac{3}{n\pi}\right)\text{Cos}\frac{n\pi x}{3}\right]_0^3$$

$$\Rightarrow b_n = \frac{2}{3}\left[\left\{-\frac{27}{n\pi}(-1)^n - \frac{54}{n^3\pi^3}(-1)^n\right\} + \frac{54}{n^3\pi^3}\right]$$

$$\Rightarrow b_n = \frac{2}{3}\left[\frac{54}{n^3\pi^3}\left\{1-(-1)^n\right\} - \frac{27}{n\pi}(-1)^n\right]$$

$$\Rightarrow b_n = \frac{2}{3}\left[\frac{108}{n^3\pi^3} + \frac{27}{n\pi}\right] \text{ where n is odd}$$

& $b_n = -\frac{18}{n\pi}$ if n is even

So Half range sine series is

$$f(x) = b_1 \text{Sin}\, x + b_2 \text{Sin}\, 2x + ...$$

$$= \frac{2}{3}\left[\frac{108}{\pi^3}\left(\frac{\text{Sin}\, x}{1^3} + \frac{\text{Sin}\, 3x}{3^3} + \frac{\text{Sin}\, 5x}{5^3} + ...\right)\right.$$

$$+ \frac{27}{\pi}\left(\frac{\text{Sin}\, x}{1} + \frac{\text{Sin}\, 3x}{3} + \frac{\text{Sin}\, 5x}{5} + ...\right)$$

$$\left. - \frac{18}{\pi}\left(\frac{\text{Sin}\, 2x}{2} + \frac{\text{Sin}\, 4x}{4} + ...\right)\right]$$

Laplace Transform

Introduction: Laplace transforms help in solving differential equations with boundary values without finding the general solution and the values of the arbitrary constants.

Definition: Let f(t) be a function defined for all positive values of t, then

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

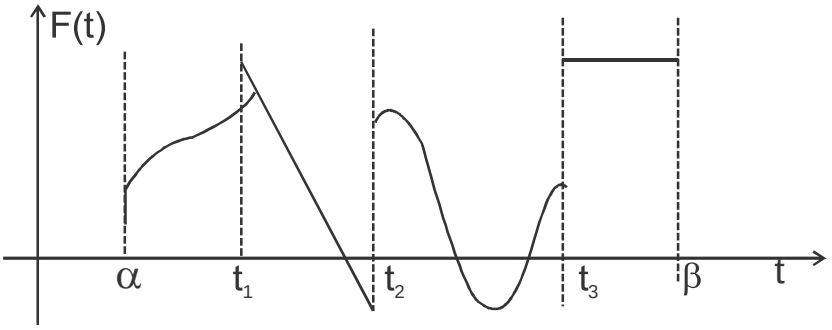
provided the integral exists, is called the Laplace Transform of f(t). It is denoted as

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt$$

Piecewise continuous function

A function $f(t)$ is said to be piecewise continuous is an interval $a \le t \le b$, if the interval can be sub divided into a finite number of intervals in each of which the function is continuous and has finite right and left hand limits.

In other words, piecewise continuous means $f(t)$ can have only finite number of finite discontinuities.



An example of a function which is periodically or sectional continuous is shown graphically in Figure above. This function has discontinuities at t_1, t_2 and t_3.

Exponential order

A function $f(t)$ is said to be exponential order if $\lim_{t\to\infty} e^{-st} f(t)$ is a finite quantity, where $s > 0$(exists).

Example: Show that the function $f(t) = e^{t^3}$ is not of exponential order.

Solution: $\lim_{t\to\infty} e^{-st}\, e^{t^3} = \lim_{t\to\infty} e^{-st+t^3} = \lim_{t\to\infty} e^{t^3-st}$

$= e^{\infty} = \infty$, not a finite quantity.

Hence $f(t) = e^{t^3}$ is not of exponential order.

Sufficient conditions for the existence of the Laplace transform

The Laplace transform of $f(t)$ exists if

i) $f(t)$ is piecewise continuous in the interval $a \leq t \leq b$

ii) $f(t)$ is of exponential order.

Note: The above conditions are only sufficient conditions and not a necessary condition.

Example: Prove that Laplace transform of $f(t) = e^{t^3}$ does not exist.

Solution: As $f(t) = e^{t^3}$ is not of exponential order. Hence Laplace transform of $f(t) = e^{t^3}$ does not exist.

<u>Important formula</u> :

(1) $L(1) = \frac{1}{s}$

(2) $L(t^n) = \frac{n!}{s^{n+1}}$ when n = 0, 1, 2, 3 .....

(3) $L(e^{at}) = \frac{1}{s-a}$ s > a

4. $L(\cosh at) = \frac{s}{(s^2 - a^2)}$ $(s^2 > a^2)$

5. $L(\sinh at) = \frac{a}{(s^2 - a^2)}$ $(s^2 > a^2)$

6. $L(\sin at) = \frac{a}{(s^2 + a^2)}$ $(s > 0)$

7. $L(\cos at) = \frac{s}{(s^2 + a^2)}$ $(s > 0)$

Proof: (1) $L(1) = \int_0^{\infty} 1.e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = -\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^{\infty}$

$$= -\frac{1}{s}(0-1) = \frac{1}{s}$$

(2) $L(t^n) = \int_0^{\infty} e^{-st}\ t^n\ dt$

Let $st = x \Rightarrow t = \frac{x}{s} \Rightarrow dt = \frac{1}{s}dx$

so $L(t^n) = \int_0^\infty e^x \left(\frac{x}{s}\right)^n \frac{dx}{s}$

or $L(t^n) = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx = \frac{n!}{s^{n+1}}$

since $\left|\overline{n+1} = \int_0^\infty e^{-x} x^n dx = n!\right.$

(3) $L(e^{at}) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)} dt$

$$= \left[\frac{e^{-(s-a)}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}(0-1) = \frac{1}{s-a}$$

(4) $L(\text{Cosh}\, at) = L\left(\frac{e^{at}+e^{-at}}{2}\right) \quad \because \text{Cosh}\, at = \frac{e^{at}+e^{-at}}{2}$

$$= \frac{1}{2}L\left(e^{at}\right) + \frac{1}{2}L\left(e^{-at}\right)$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)}\right] = \frac{s}{s^2-a^2}$$

(5) $L(\text{Sin h}\, at) = L\left[\frac{1}{2}e^{at} - e^{-at}\right]$

$$= \frac{1}{2}\left[L\left(e^{at}\right) - L\left(e^{-at}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a-s+a}{(s-a)(s+a)}\right] = \frac{a}{s^2-a^2}$$

(6) $L(\text{Sin}\, at) = L\left(\frac{e^{iat}-e^{-iat}}{2i}\right) \quad \because \text{Sin}\, at = \frac{e^{iat}-e^{-iat}}{2}$

$$= \frac{1}{2i}\left[L\left(e^{iat} - e^{-iat}\right)\right]$$

$$= \frac{1}{2i}\left[L\left(e^{iat}\right) - L\left(e^{-iat}\right)\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right]$$

$$= \frac{1}{2i}\left[\frac{s+ia-s+ia}{s^2+a^2}\right] = \frac{a}{s^2+a^2}$$

Problem: Find the laplace transform of (1 + Cos 2t)

Solution: Laplace transform of (1 + Cos 2t) dt

$$= \int_0^\infty e^{-st}(1+\text{Cos}\,2t)dt$$

$$e = \int_0^\infty e^{-st}\left(1+\frac{e^{2it}+e^{-2it}}{2}\right)dt$$

$$= \frac{1}{2}\int_0^\infty \left[2e^{-st} + e^{(-s+2i)} + e^{(-s-2i)t}\right]dt$$

$$= \frac{1}{2}\left[\frac{2e^{-st}}{-s} + \frac{e^{(-s+2i)t}}{-s+2i} + \frac{e^{(-s-2i)t}}{-s-2i}\right]_0^\infty$$

$$= \frac{1}{2}\left[\left(0+\frac{2}{s}\right) + \frac{1}{-s+2i}(0-1) + \frac{1}{-s-2i}(0-1)\right]$$

$$= \frac{1}{2}\left[\frac{2}{s} + \frac{1}{s-2i} + \frac{1}{s+2i}\right] = \frac{1}{2}\left[\frac{2}{s} + \frac{2s}{s^2+4}\right]$$

$$= \frac{1}{s} + \frac{s}{s^2+4} = \frac{2s^2+4}{s(s^2+4)}$$

Properties of Laplace Transform:

Linear Property:

$$L\left[af_1(t)+bf_2(t)\right] = aLf_1(t)+bLf_2(t)$$

Proof : $L\left[af_1(t)+bf_2(t)\right] = \int_0^\infty e^{-st}\left[af_1(t)+bf_2(t)\right]$

$$= a\int_0^\infty e^{-st} f_1(t)dt + b\int_0^\infty e^{-st} f_2(t)dt$$

$$= aL\left[f_1(t)\right] + bL\left[f_2(t)\right]$$

First shifting theorem:

If $Lf(t) = F(s)$, then

$$L\left[e^{at} f(t)\right] = F(s-a)$$

Proof : $L\left[e^{at} f(t)\right] = \int_0^\infty e^{-st} e^{at} f(t)dt$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= F(s-a)$$

By this property we have

(1) $L(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}} \because L(t^n) = \frac{n!}{s^{n+1}}$

(2) $L(e^{at} \operatorname{Cosh} bt) = \frac{s-a}{(s-a)^2 - b^2}$

(3) $L(e^{at} \operatorname{Sinh} bt) = \frac{b}{(s-a)^2 - b^2}$

(4) $L(e^{at} \operatorname{Sin} bt) = \frac{b}{(s-a)^2 + b^2}$

(5) $L(e^{at} \operatorname{Cos} bt) = \frac{s-a}{(s-a)^2 + b^2}$

Problem: Find the Laplace transform of $\operatorname{Cos}^2 t$

Solution : $\operatorname{Cos} 2t = \operatorname{Cos}^2 t - 1$

$$\operatorname{Cos}^2 t = \frac{1}{2}\left(\operatorname{Cos} 2t + 1\right)$$

$$L\left(\operatorname{Cos}^2 t\right) = L\left[\frac{1}{2}\left(\operatorname{Cos} 2t + 1\right)\right]$$

$$= \frac{1}{2}\left[\left(L \operatorname{Cos} 2t + L(1)\right)\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2 + 2^2} + \frac{1}{s}\right] = \frac{1}{2}\left[\frac{s}{s^2 + 4} + \frac{1}{s}\right]$$

Problem : Find the Laplace transform of $t^2 \operatorname{Cos} at$

Solution : $L\left(t^2 \operatorname{Cos} at\right) = L\left[t^2 . \frac{e^{at} + e^{-at}}{2}\right]$

$$= \frac{1}{2}\left[L\left(t^2 e^{iat}\right) + L\left(t^2 e^{-iat}\right)\right]$$

$$= \frac{1}{2}\left[\frac{2!}{(s-ia)^3} + \frac{2!}{(s+ia)^3}\right]$$

$$= \frac{(s+ia)^3 + (s-ia)^3}{(s-ia)^3.(s+ia)^3}$$

$$= \frac{(s^3 + 3ias^2 - 3a^2s - ia^3) + (s^3 - 3ias^2 - 3a^2s + ia^3)}{(s^2 - a^2)^3}$$

$$= \frac{2s^3 - 6a^2s}{(s^2 - a^2)^3} = \frac{2s(s^2 - 3a^2)}{(s^2 + a^2)^3}$$

Problem : Find the Laplace transform of $t \operatorname{Sin} at$

Solution : $L(t \operatorname{Sin} at) = L\left[t.\frac{e^{at} - e^{-at}}{2i}\right]$

$$= \frac{1}{2i}\left[L\left(te^{iat}\right) - L\left(te^{-iat}\right)\right]$$

$$= \frac{1}{2i}\left[\frac{1}{(s-ia)^2} - \frac{1}{(s+ia)^2}\right]$$

$$= \frac{1}{2i}\left[\frac{(s+ia)^2 - (s-ia)^2}{(s+ia)^2.(s-ia)^2}\right]$$

$$= \frac{1}{2i}\frac{(s^2 + 2ias - a^2) - (s^2 - 2ias - a^2)}{(s^2 + a^2)^2}$$

$$= \frac{1}{2i}\frac{4ias}{(s^2 + a^2)^2} = \frac{2as}{(s^2 + a^2)^2}$$

Initial Value theorem: Let L (f(t) = F (s) then

$$\lim f(t)_{t \longrightarrow 0} = \lim f(s)_{s \longrightarrow \infty}$$ provided the limit exists.

Final Value Theorem : Let $L[f(t)] = F(s)$

then $\lim_{t \to \infty} f(t) = \lim_{s \to 0}[sF(s)]$ provided this limit exists.

Laplace transform of derivative of f (t)

$$L[f'(t)] = sL[f'(t)] - f(0) \quad where \quad L[f(t)] = F(s)$$

Proof : $L[f'(t)] = \int_0^\infty e^{-st} f(t)dt$

$$= \left[e^{-st} f(t) \right]_0^{\infty} - \int_0^{\infty} (-se^{-st}) f(t) dt$$

$$= -f(0) + s \int_0^{\infty} e^{-st} f(t) dt \quad \left[e^{-st} f(t) = 0 \ \ as \ \ F \to \infty \right]$$

$$= -f(0) + sLf(t)$$

or $L[f'(t)] = sL[f'(t)] - f(0)$

Laplace transform of derivatives of order n.

$$L\left[f^n(t) \right] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) ..... - f^{n-1}(0)$$

Proof : We know $L[f'(t)] = sL[f(t)] - f(0)$

Replacing $f(t)$ *by* $f'(t)$

$$L[f''(t)] = sL[f'(t)] - f'(0)$$

$$= s[sLf(t) - f(0)] - f'(0)$$

$$= s^2 Lf(t) - sf(0) - f'(0)$$

similarly $L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$

$$L\left[f^{iv}(t) \right] = s^4 L[f(t)] - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0)$$

.....

$$L\left[f^n(t) \right] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) ..... - f^{n-1}(0)$$

Transform of Integral of f(t) :

$$L\left[\int_0^t f(t) dt \right] = \frac{1}{s} F(s) \ \ where \ \ L[f(t)] = F(s)$$

Proof : Let $\phi(t) = \int_0^t f(t) dt$ *and* $\phi(0)$ *then* $p'(t) = f(t)$

and $L[\phi'(t)] = sL[\phi(t)] - \phi(0)$

$$= sL[\phi(t)]$$

so $L[\phi(t)] = \frac{1}{s} L[\phi'(t)]$

Putting the values of $\phi(t)$ and $\phi'(t)$, we get

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$$

or $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$

Laplace transform of t. f(t) [Multiplication by t]

If $L[f(t)] = F(s)$, then

$$L[t^n f(t)] = (-1)^n \left(\frac{d}{ds}\right)^n F(s)$$

Proof : $L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt$

Differentiating with respect to s, we get

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)dt\right]$$

$$= \int_0^\infty \frac{d}{ds} e^{-st} f(t)dt$$

$$= \int_0^\infty \left(-te^{-st}\right).f(t)dt$$

$$= \int_0^\infty e^{-st}\left[-t.f(t)\right]dt$$

$$= L[-t.f(t)]$$

or $L[tf(t)] = (-1)^1 \frac{d}{ds}[F(s)]$

similarly, $L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}[F(s)]$

$$L[t^3 f(t)] = (-1)^3 \frac{d^3}{ds^3}[F(s)]$$

.....

.....

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Problem: Find the Laplace transform of t sinh at.

Solution: $L[t\sinh at] = -\frac{d}{ds}\left[\frac{a}{(s^2-a^2)}\right] = \frac{2as}{(s^2-a^2)^2}$.

Problem: Find the Laplace transform of t^2 cos at.

Solution: $L[t^2\cos at] = \frac{d^2}{ds^2}\frac{s}{(s^2+a^2)^2} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$.

Laplace transform of $\frac{1}{t}f(t)$ [Division by t]

Th. If $L[f(t)] = F(s)$ then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)ds$

Proof : $L[f(t)] = F(s) = \int_0^\infty e^{-st}f(t)dt$

Integrating with respect to *s* we get

$$\int_s^\infty F(s)ds = \int_s^\infty\left[\int_0^\infty e^{-st}f(t)dt\right]ds$$

$$= \int_s^\infty\left[\int_s^\infty e^{-st}f(t)ds\right]dt$$

Interchanging the limits

$$= \int_0^\infty -\frac{f(t)}{t}\left[e^{-st}\right]_s^\infty dt$$

$$= \int_0^\infty -\frac{f(t)}{t}\left[0-e^{-st}\right]dt$$

$$= \int_0^\infty e^{-st}\left[\frac{1}{t}f(t)dt\right]dt$$

$$= L\left[\frac{1}{t}f(t)\right]$$

or $= L\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)ds$

Problem : Find the Laplace transform of $\mathrm{Sin}\frac{2t}{t}$

Solution : $L(\mathrm{Sin}\,2t) = \frac{2}{s^2+4}$

so $L\left(\frac{\text{Sin}\,2t}{t}\right)=\int_s^\infty \frac{2}{s^2+4}ds = 2.\frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty$

$$=\tan^{-1}\infty-\tan^{-1}\frac{s}{2}$$

$$=\frac{\pi}{2}-\tan^{-1}\frac{s}{2}$$

$$=Cot^{-1}\frac{s}{2}$$

Problem : If $f(t)=\frac{e^{at}-\text{Cos}\,bt}{t}$, find the Laplace transform of f (t).

Solution : $f(t)=\frac{e^{at}-\text{Cos}\,bt}{t}$

we know that, $L\left(e^{at}-\text{Cos}\,bt\right)=\left(\frac{1}{(s-a)}-\frac{s}{\left(s^2-b^2\right)}\right)$

so $L\left(\frac{e^{at}-\text{Cos}\,bt}{t}\right)=\int_s^\infty\left(\frac{1}{(s-a)}-\frac{s}{\left(s^2-b^2\right)}\right)ds$

$$=\left[\log(s-a)-\frac{1}{2}\log\left(s^2-b^2\right)\right]_s^\infty$$

$$=\left[\frac{\log(s-a)-\log\left(s^2-b^2\right)}{2}\right]_s^\infty$$

$$=\frac{1}{2}\left[\log(s-a)-\log\left(s^2-b^2\right)\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\frac{(s-a)^2}{\left(s^2+b^2\right)^2}\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\left\{\frac{\left(1-\frac{a}{s}\right)^2}{\left(1+\frac{b^2}{s^2}\right)^2}\right\}\right]_s^\infty=\frac{1}{2}\left[0-\log\left\{\frac{\left(1-\frac{a}{s}\right)^2}{\left(1+\frac{b^2}{s^2}\right)^2}\right\}\right]$$

$$=\frac{1}{2}\left[\log\frac{\left(s^2+b^2\right)}{(s-a)^2}\right]$$

Problem: Find the Laplace transform of $\frac{(1-\cos t)}{t^2}$.

Solution : $L\left[\frac{1-\cos t}{t^2}\right] = \cot^{-1} s + \left(\frac{s}{2}\right) \log\left[\frac{s^2}{(s^2+1)}\right]$.

Unit Step Function: The unit step function 4 (t - a) is defined as.

$$u(t-a) = \begin{cases} 0 & when \quad t < a \\ 1 & when \quad t \geq a \end{cases}$$

Laplace Transform of Unit Step Function:

$$L[u(t-a)] = \frac{e^{-as}}{s}$$

Proof : $L[u(t-a)] = \int_0^\infty e^{-as} u(t-a)dt$

$$= \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}.1dt$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$\therefore L[u(t-a)] = \frac{e^{-as}}{s}$$

Problem: Express the following function in terms of unit step function and find its Laplace transform

$$f(t) = \begin{cases} 8 & t < 2 \\ 6 & t \geq 2 \end{cases}$$

Solution : $f(t) = \begin{cases} 8+0 & t < 2 \\ 6-2 & t \geq 2 \end{cases}$

$$= 8 + \begin{cases} 0 & t < 2 \\ -2 & t \geq 2 \end{cases} = 8 + (-2)\begin{cases} 0 & t < 2 \\ 1 & t \geq 2 \end{cases} = 8 - 2u(t-2) - 2s$$

so $Lf(t) = 8L(1) - 2L4(t-2) = \frac{8}{s} - 2\frac{e}{s}$

Second Shifting theorem:

If $L[f(t)] = F(s)$ *then* $L[f(t-a)u(t-a)] = e^{-as}F(s)$

Proof : $L[f(t-a)u(t-a)] = \int_0^\infty e^{-st}[f(t-a)u(t-a)]dt$

$$= \int_0^\infty e^{-st} f(t-a).0dt + \int_0^\infty e^{-st} f(t-a).1dt$$

$$= \int_a^\infty e^{-st} f(t-a)dt$$

Now let so $t-a=u \Rightarrow dt=du$

Let $u=t-a$ so $= \int_0^\infty e^{-s(u+a)} f(u)du$

$$= e^{-sa}\int_0^\infty e^{-su} f(u)du = e^{-as}F(s)$$

Theorem : $L\left[f(t)u(t-a)\right] = e^{-as}L\left[f(t+a)\right]$

Proof : $L\left[f(t)u(t-a)\right] = \int_0^\infty e^{-st}\left[f(t)u(t-a)\right]dt$

$$= \int_0^a e^{-st} f(t).0dt + \int_0^a e^{-st} f(t).(1)dt$$

$$= 0 + \int_a^\infty e^{-st} f(t)dt$$

Let $t-a=y \Rightarrow t=a+y$

$$= \int_a^\infty e^{-s(y+a)}.f(y+a)dy$$

$$= e^{-as}\int_a^\infty e^{-sy} f(y+a)dy$$

$$= e^{-as}\int_a^\infty e^{-st} f(t+a)dt$$

$$= e^{-as}Lf(t+a)$$

Problem: Express the following function in terms of unit step function and find its Laplace transform.

Solution : Given function in algebraic form may be written as $f(t) = \begin{cases} 0 & 0<t<1 \\ t-1 & 1<t<2 \\ 1 & 2<t \end{cases}$

or

$$f(t) = (t-1)\left[u(t-1) - u(t-2)\right] + 1.u(t-2)$$

$$= (t-1)u(t-1) - u(t-2)\left\{t-1-1\right\}$$

$$= (t-1)u(t-1) - u(t-2)u(t-2)$$

So $Lf(t) = L(t-1)u(t-1) - L(t-2)u(t-2) = \frac{e^{-s}}{s^2} - L(t-2)u(t-2) = \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}$

Problem: Express the following function in terms of unit step function.

$$f(t) = \begin{cases} t-1 & 1<t<2 \\ 3\text{-t} & 2<t<3 \end{cases}$$

and find it Laplace' transform.

Solution : $f(t) = \begin{cases} t-1 & 1<t<2 \\ 3\text{-t} & 2<t<3 \end{cases}$

$$= (t-1)\left[u(t-1) - u(t-2)\right] + (3-t)\left[u(t-2) - u(t-3)\right]$$

$$= (t-1)u(t-1) - 2(t-2) + (t-3)u(t-3)$$

so $Lf(t) = \frac{e^{-s}}{s^2} - 2\frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$

<u>Change of Scale Property</u> : If $Lf(t) = F(s)$ *then* $L\left[f(at)\right] = \frac{1}{a}F\left(\frac{s}{a}\right)$

Proof : $Lf(t) = \int_0^\infty e^{-st} f(at)dt = \int_0^\infty e^{-st}.f(u)\frac{du}{a}$ *put* $at = u \Rightarrow dt = \frac{du}{a}$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}u}.f(u)du = \frac{1}{a}F\left(s/a\right)$$

<u>Periodic Function</u>: Let f(t) be a periodic function with period T, then

$$L\left[f(t)\right] = \frac{\int_0^T e^{-st} f(t)dt}{1-e^{-sT}}$$

Proof : $L\left[f(t)\right] = \int_0^T e^{-st} f(t)dt$

$$= \int_0^T e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \int_{2T}^{3T} e^{-st} f(t)dt + ...$$

Substituting t = u + T in second integral and t = u + 2T in third integral and so on

$$L[f(t)] = \int_0^T e^{-st} f(t)dt + \int_0^T e^{-s(u+t)} f(u+t)du + \int_0^T e^{-s(u+2T)} f(u+2t)du + ...$$

$$= \int_0^T e^{-st} f(t)dt + e^{-st}\int_0^T e^{-st} f(u)du + e^{-2sT}\int_0^T e^{-su} f(u) + ...$$

$$f(u) = f(u+T) = f(u+2T) = f(u+3T) = ...$$

$$= \int_0^T e^{-st} f(t)dt + e^{-st}\int_0^T e^{-st} f(t)dt + e^{-2sT}\int_0^T e^{-st} f(t) + ...$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$$

$$\text{as } \left[1 + a + a^2 + a^3 + ... = \frac{1}{1-a}\right]$$

Problem: Find the Laplace transform of the wave form

$$f(t) = \left(\frac{2t}{3}\right), 0 \le t \le 3$$

Solution : $L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$

$$= \frac{1}{1-e^{-3s}}\int_0^3 e^{-st}\left(\frac{2t}{3}\right)dt$$

$$= \frac{1}{1-e^{-3s}} \times \frac{2}{3}\left[\frac{te^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right]_0^3$$

$$= \frac{2}{3}\frac{1}{1-e^{-3s}}\left[\frac{3e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right]$$

$$= \frac{2}{3}\frac{1}{1-e^{-3s}}\left[\frac{3e^{-3s}}{-s} + \frac{1-e^{-3s}}{s^2}\right]$$

$$= \frac{2e^{-3s}}{-s\left(1-e^{-3s}\right)} + \frac{2}{2s^2}$$

Problem: Find the Laplace transform of the function (Half wave rectifier)

$$f(t) = \begin{cases} \text{Sin } wt & \text{for } 0 < t < \pi / w \\ 0 & \text{for } \pi/w < t < 2\pi / w \end{cases}$$

Solution : $L\left[f(t)\right]=\dfrac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$

$$=\frac{1}{1-e^{\frac{-2\pi s}{w}}}\int_0^{2\pi/w} e^{-st}f(t)dt$$

f(t) is a periodic function with period $T=\dfrac{2\pi}{w}$

$$=\frac{1}{1-e^{\frac{-2\pi s}{w}}}\int_0^{\pi/w} e^{-st}\operatorname{Sin} wtdt+\int_{\pi/w}^{2\pi/w} e^{-st}\times 0\times dt$$

$$=\frac{1}{1-e^{\frac{-2\pi s}{w}}}\left[\int_0^{\pi/w} e^{-st}\operatorname{Sin} wtdt\right]$$

as $=\int e^{ax}\operatorname{Sin} bxdx=\dfrac{e^{ax}(a\operatorname{Sin} bx-b\operatorname{Cos} bx)}{a^2+b^2}$

so $=\dfrac{1}{1-e^{\frac{-2\pi s}{w}}}\left[e^{-st}\dfrac{\left(-s\operatorname{Sin} wt-w\operatorname{Cos} wt\right)}{s^2+w^2}\right]_0^{\pi/w}$

$$=\frac{1}{1-e^{\frac{-2\pi s}{w}}}\left[\frac{we^{\frac{-\pi s}{w}}+w}{s^2+w^2}\right]$$

$$=\frac{w\left[1+e^{\frac{-\pi s}{w}}\right]}{\left(s^2+w^2\right)\left[1-e^{\frac{-2\pi s}{w}}\right]}$$

$$=\frac{w}{\left(s^2+w^2\right)\left[1-e^{\frac{-\pi s}{w}}\right]}$$

Problem: Find the Laplace transform of the function.

$$f(t)=te^{-t}\operatorname{Sin} 2t$$

Solution : $L\left(\operatorname{Sin} 2t\right)=\dfrac{2}{s^2+4}$

$$L\left(e^{-t}\operatorname{Sin} 2t\right)=\frac{2}{(s+1)^2+4}F(s)(say)$$

$$L\left(te^{-t}\operatorname{Sin} 2t\right)=-\frac{d}{ds}\left[F(s)\right]=-\frac{d}{ds}\left[\frac{2}{(s+1)^2+4}\right]$$

$$= \frac{4(s+1)}{\left[(s+1)^2+4\right]^2}$$

Convolution Theorem:

If $L[f_1(t)] = F_1(s)$ *and* $L[f_2(t)] = F_2(s)$

then $L\left\{\int_0^t f_1(x) f_2(t-x)dx\right\} = F_1(s).F_2(s)$

Unit impulse function: when a large force acts for a short time, then the product of the force and time is called impulse (in applied mechanics)

Unit impulse function is the limiting function & $\delta(t-1) = \begin{cases} \frac{1}{\in}, & a < t < a+\in \\ 0 & otherwise \end{cases}$

The value of function becomes infinite as $\in \to 0$ and the area of rectangle is unity.

The unit Impulse function is defined as $\delta(t-1) = \begin{cases} \infty & for \ \ t = a \\ 0 & for \ \ t \neq a \end{cases}$ with $\int_0^\infty s(t-1)dt = 1$

Laplace Transform of Unit Impulse Function: $L[\delta(t-a)\ ] = \int_0^\infty f(t)\delta(t-a)dt = e^{-as}$

$L[\delta(t)] = e^{(0)} = 1$

Problem: Find the Laplace transform of (a) $\int_{-\infty}^{+\infty} e^{-st}\delta(t-2)$

(b) $t^3\delta(t-4)$

Solution : (a) $\int_{-\infty}^{+\infty} e^{-5t}\delta(t-2) = e^{-5\times 2} = e^{-10}$

(b) $Lt^3\delta(t-4) = \int_0^\infty e^{-st}t^3\delta(t-4)dt = 4^3e^{-4s}$

Inverse Laplace transform:

If We find f(t) when F(s) is given, is then inverse Laplace transform of F(s) is f(t).

if $L[f(t)] = F(s)$ *then* $L^{-1}[F(s)] = f(t)$

Formula to find inverse Laplace transform:

(1) $L^{-1}(1/s) = 1.$

(2) $L^{-1}(1/s^n) = t^{(n-1)}/(n-1)!.$

(3) $L^{-1}\big(1/(s-a)\big) = e^{(at)}.$

(4) $L^{-1}\big(s/(s^2-a^2)\big) = \cosh at.$

(5) $L^{-1}\big(1/(s^2-a^2)\big) = (1/a)\sinh at.$

(6) $L^{-1}\big(1/(s^2+a^2)\big) = (1/a)\sin at.$

(7) $L^{-1}\big(s/(s^2+a^2)\big) = \cos at.$

(8) $L^{-1}[F(s-a)] = e^{(at)}f(t).$

(9) $L^{-1}(1/[(s-a)^2+b^2]) = (1/b)e^{(at)}\sin bt.$

(10) $L^{-1}\big((s-a)/[(s-a)^2+b^2]\big) = e^{(at)}\cos bt.$

(11) $L^{-1}(1/[(s-a)^2-b^2]) = (1/b)e^{(at)}\sinh bt.$

(12) $L^{-1}\big((s-a)/[(s-a)^2-b^2]\big) = e^{(at)}\cosh bt.$

(13) $L^{-1}(1/(s^2+a^2)^2) = (1/2a^3)(\sin at - at\cos at).$

(14) $L^{-1}(s/(s^2+a^2)^2) = (1/2a)t\sin at.$

(15) $L^{-1}\big((s^2-a^2)/(s^2+a^2)^2\big) = t\cos at.$

(16) $L^{-1}(1) = \delta(t).$

(17) $L^{-1}(s^2/(s^2+a^2)^2) = (1/2a)[\sin at + at\cos at].$

Problem: Find the inverse Laplace Transform of the following:

(i) $1/(s-2)$ Ans. $e^{(2t)}$

(ii) $1/(s^2-9)$ Ans. $(1/3)\sinh 3t$

(iii) $s/(s^2-16)$ Ans. $\cosh 4t$

(iv) $1/(s^2+25)$ Ans. $(1/5)\sin 5t$

(v) $s/(s^2+9)$ Ans. $\cos 3t$

Problem: Find the inverse Laplace transform of:

1. $(3s-8)/(4s^2+25)$ Ans. $(3/4)\cos(5t/2)-(4/5)\sin(5t/2)$

2. $3(s^2-2)^2/(2s^5)$ Ans. $(3/2)-3t^2+(1/2)t^4$

3. $(2s-5)/(4s^2+25)+(4s-18)/(9-s^2)$

Ans. $(1/2)[\cos(5t/2)-\sin(5t/2)]-4\cosh 3t+6\sinh 3t$

4. $(5s-10)/(9s^2-16)$ Ans. $(5/9)\cosh(4t/3)-(5/6)\sinh(4t/3)$

5. $(1/4s)+16/(1-s^2)$ Ans. $(1/4)-16\sinh t$

6. $1/[(s-2)^2+1]$ Ans. $e^{(2t)}\sin t$

7. $(s-1)/[(s-1)^2+4]$ Ans. $e^t\cos 2t$

8. $1/[(s+3)^2-4]$ Ans. $(1/2)e^{(-3t)}\sinh 2t$

9. $(s+2)/[(s+2)^2-25]$ Ans. $e^{(-2t)}\cosh 5t$

10. $1/(2s-7)$ Ans. $(1/2)e^{(7t/2)}$

Multiplication by s: $L^{-1}[sF(s)]=(d/dt)f(t)+f(0)\delta(t)$

Problem: Find the inverse Laplace transform of:

(i) $s/(s^2+1)$ Ans. $\cos t$

(ii) $s/(4s^2-25)$ Ans. $(1/4)\cosh(5t/2)$

(iii) $3s/(2s+9)$ Ans. $-(27/4)e^{(-11t/2)}+3/2$

Division by s (multiplication by 1/s)

$L^{-1}[F(s)/s]=\int_0^t L^{-1}[F(s)]\,dt=\int_0^t f(t)\,dt.$

Problem: Find the inverse Laplace transform of:

(i) $1/[s(s+a)]$ Ans. $(1/a)\left[1-e^{(-at)}\right]$

(ii) $1/[s(s^2+1)]$ Ans. $1-\cos t$

(iii) $(s^2+3)/[s(s^2+9)]$ Ans. $(1/3)[2\cos 3t+1]$

First Shifting Property:

If $L^{-1}F(s)=f(t)$ *then* $L^{-1}F(s+a)=e^{-at}L^{-1}F(s)$

Problem : Find Inverse Laplace transform of $\frac{1}{(s+2)^5}$

Solution : $L^{-1}\frac{1}{s^5}=\frac{t^4}{4!}$

so $L^{-1}\frac{1}{(s+2)^5}=e^{-2t}\times\frac{t^4}{4!}$

Second Shifting Property:

$$L^{-1}\left[e^{-as}F(s)\right]=f(t-a)U(t-a)$$

Problem : Evaluate $L^{-1}\left[\frac{e^{-s}-3e^{-3s}}{s^2}\right]$

Solution : $L^{-1}\left[\frac{e^{-s}-3e^{-3s}}{s^2}\right]=L^{-1}\left[\frac{e^{-s}}{s^2}-\frac{3e^{-3s}}{s^2}\right]$

we know that $Lu(t-a)=\frac{e^{-as}}{s}$ and $L\left[(t-a)u(t-a)\right]=\frac{e^{-as}}{s^2}$

so $L^{-1}\left[\frac{e^{-s}-3e^{-3s}}{s^2}\right]=(t-1)u(t-1)-3(t-3)u(t-3)$

Problem : Obtain inverse Laplace transform of $\frac{e^{-\pi s}}{(s+3)}$

Solution : $L^{-1}\left(\frac{1}{(s+3)}\right)=e^{-3t}$ so $L^{-1}\left(\frac{e^{-\pi s}}{(s+3)}\right)=e^{-3(t-\pi)}U(t-\pi)$.

Inverse Laplace transform of derivatives:

Since $L\left[tf(t)\right]=-1\frac{d}{ds}F(s)$ so $L^{-1}\left[\frac{d}{ds}F(s)\right]=-tL^{-1}F(s)$

or $L^{-1}\left[F(s)\right]=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$

Problem : Find the inverse Laplace transform of $\tan^{-1}\frac{1}{s}$

Solution : $L^{-1}\left(\tan^{-1}\frac{1}{s}\right)=-\frac{1}{t}L^{-1}\left(\frac{d}{ds}\tan^{-1}\frac{1}{s}\right)=-\frac{1}{t}L^{-1}\left(\frac{1}{1+\frac{1}{s^2}}\times\left(-\frac{1}{s^2}\right)\right)$

$$=+\frac{1}{t}L^{-1}\left(\frac{1}{1+\frac{1}{s^2}}\right)=\frac{\sin t}{t}.$$

Problem : Find inverse Laplace transform of $F(s)=\log\frac{s+a}{s+b}$

Solution : $L^{-1}\log\left(\frac{s+a}{s+b}\right)=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\left(\frac{s+a}{s+b}\right)\right]$

$$=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log(s+a)-\frac{d}{ds}\log(s+a)\right]$$

$$=-\frac{1}{t}L^{-1}\left[\frac{1}{(s+a)}-\frac{1}{(s+a)}\right]$$

$$=-\frac{1}{t}\left[e^{-a+}-e^{-b+}\right]$$

<u>Inverse Laplace transform of Integrals</u>:

$$L^{-1}\left[\int_s^\theta F(s)ds\right]=\frac{f(t)}{t}=\frac{1}{t}L^{-1}\left[F(s)\right] \text{ or } L^{-1}\left[F(s)\right]=tL^{-1}\left[\int_s^\theta F(s)ds\right]$$

Problem : Obtain $L^{-1}\frac{2s}{\left(s^2+1\right)^2}$

Solution : $L^{-1}\frac{2s}{\left(s^2+1\right)^2}=tL^{-1}\int_s^\theta\frac{2sds}{\left(s^2+1\right)^2}=tL^{-1}\left[-\frac{1}{s^2+1}\right]_s^\theta$

$$=tL^{-1}\left[-0+\frac{1}{s^2+1}\right]=tL^{-1}\left[\frac{1}{s^2+1}\right]=t.\sin t.$$

Problem: Find the inverse Laplace transforms of $\frac{(s+4)}{s(s-1)(s^2+4)}$.

Solution: Partial fraction decomposition

$$\frac{(s+4)}{s(s-1)(s^2+4)}=-\frac{1}{s}+\frac{1}{(s-1)}-\frac{1}{(s^2+4)}$$

$$\therefore \mathrm{L}^{-1}\left[\frac{(\mathrm{s}+4)}{\mathrm{s}(\mathrm{s}-1)(\mathrm{s}^2+4)}\right] = \mathrm{L}^{-1}\left[-\frac{1}{\mathrm{s}}+\frac{1}{(\mathrm{s}-1)}-\frac{1}{(\mathrm{s}^2+4)}\right]$$

$$= -\mathrm{L}^{-1}[1/\mathrm{s}] + \mathrm{L}^{-1}[1/(\mathrm{s}-1)] - (1/2)\mathrm{L}^{-1}[2/(\mathrm{s}^2+2^2)]$$

$$= -1 + \mathrm{e}^{\mathrm{t}} - (1/2)\sin 2\mathrm{t}$$

Problem: Find the Laplace inverse of $\mathrm{s}^2/[(\mathrm{s}^2+\mathrm{a}^2)(\mathrm{s}^2+\mathrm{b}^2)]$.

Solution. Let us convert the given function into partial fractions.

$$\mathrm{L}^{-1}[\mathrm{s}^2/(\mathrm{s}^2+\mathrm{a}^2)(\mathrm{s}^2+\mathrm{b}^2)]$$

$$= \mathrm{L}^{-1}[\mathrm{a}^2/(\mathrm{a}^2-\mathrm{b}^2)\cdot 1/(\mathrm{s}^2+\mathrm{a}^2) - \mathrm{b}^2/(\mathrm{a}^2-\mathrm{b}^2)\cdot 1/(\mathrm{s}^2+\mathrm{b}^2)]$$

$$= 1/(\mathrm{a}^2-\mathrm{b}^2)\mathrm{L}^{-1}[\mathrm{a}^2/(\mathrm{s}^2+\mathrm{a}^2) - \mathrm{b}^2/(\mathrm{s}^2+\mathrm{b}^2)]$$

$$= 1/(\mathrm{a}^2-\mathrm{b}^2)\left[\mathrm{a}^2\left((1/\mathrm{a})sinat\right) - \mathrm{b}^2\left((1/\mathrm{b})\sin\mathrm{bt}\right)\right]$$

$$= 1/(\mathrm{a}^2-\mathrm{b}^2)[\mathrm{asinat} - \mathrm{bsinbt}]$$

<u>Inverse Laplace Transform by Convolution</u>:

$$L\left\{\int_0^t f_1(x)\, f_2(t-x)dx\right\} = f_1(s).f_2(s)$$

or $L^{-1}\left[f_1(s).f_2(s)\right] = \int_0^t f_1(x)\, f_2(t-x)dx$

Problem : Evaluate $L^{-1}\left\{\frac{s}{\left(s^2+1\right)\left(s^2+4\right)}\right\}$

Solution: By Convolution theorem

$$L^{-1}\left\{\frac{s}{\left(s^2+1\right)\left(s^2+4\right)}\right\} = \frac{1}{2}L^{-1}\left[\frac{s}{\left(s^2+1\right)}\times\frac{2}{\left(s^2+4\right)}\right]$$

$$= \frac{1}{2}\int_0^t \mathrm{Sin}\,2x\,\mathrm{Cos}(t-x)dx$$

$$= \int_0^t \mathrm{Sin}\,x\,\mathrm{Cos}\,x\left\{\mathrm{Cos}\,t\,\mathrm{Cos}\,x + \mathrm{Sin}\,t\,\mathrm{Sin}\,x\right\}dx$$

$$= \int_0^t \left\{\mathrm{Sin}\,x\,\mathrm{Cos}^2\,x\,\mathrm{Cos}\,t + \mathrm{Sin}^2\,x\,\mathrm{Cos}\,x\,\mathrm{Sin}\,t\right\}dx$$

$$= \left[-\frac{\mathrm{Cos}^3\,x}{3}\mathrm{Cos}\,t + \frac{\mathrm{Sin}^3\,x}{3}\mathrm{Sin}\,t\right]_0^t$$

$$= -\frac{\text{Cos}^4 t}{3} + \frac{\text{Sin}^4 t}{3} + \frac{\text{Cos}\, t}{3}$$

$$= \frac{1}{3}\left[\text{Sin}^4 t - \text{Cos}^4 t\right] + \frac{\text{Cos}\, t}{3}$$

$$= \frac{1}{3}\left(\text{Sin}^2 t + \text{Cos}^2 t\right)\left(\text{Sin}^2 t - \text{Cos}^2 t\right) + \frac{\text{Cos}\, t}{3}$$

$$= \frac{1}{3}\left(\text{Sin}^2 t - \text{Cos}^2 t\right) + \frac{\text{Cos}\, t}{3}$$

$$= -\frac{1}{3}\text{Cos}\, 2t + \frac{\text{Cos}\, t}{3}$$

$$= \frac{1}{3}\left(\text{Cos}\, t - \text{Cos}\, 2t\right)$$

Problem: Using the convolution theorem, find $L^{-1}[s^2/(s^2 + a^2)(s^2 + b^2)], a \neq b$.

Solution. We have$L(\text{cosat}) = s/(s^2 + a^2)$ and $L(\text{cosbt}) = s/(s^2 + b^2)$

Hence by the convolution theorem

$$L\int_0^t \text{cosax}\ \text{cosb}(t - x)dx = s^2/[(s^2 + a^2)(s^2 + b^2)]$$

Therefore, $L^{-1}[s^2/(s^2 + a^2)(s^2 + b^2)] = \int_0^t \text{cosax}\ \text{cosb}(t - x)dx$

$$= (1/2)\int_0^t [\cos(ax + bt - bx) + \cos(ax - bt + bx)]\, dx$$

$$= (1/2)\int_0^t \cos[(a - b)x + bt]dx + (1/2)\int_0^t \cos[(a + b)x + bt]dx$$

$$= [\sin[(a - b)x + bt]/2(a - b)]_0{}^t + [\sin[(a + b)x - bt]/2(a + b)]_0{}^t$$

$$= (\text{sinat} - \text{sinbt})/[2(a - b)] + (\text{sinat} + \text{sinbt})/[2(a + b)]$$

$$= (\text{asinat} - \text{bsinbt})/(a^2 - b^2)$$

Problem: Obtain $L^{-1}[1/s(s^2 + a^2)]$.

Solution. $L^{-1}[1/s] = 1$ and $L^{-1}[1/(s^2 + a^2)] = (\text{sinat})/a$

Hence by the convolution theorem

$$L\int_0^t 1 \cdot \frac{\text{sina}(t - x)}{a}dx = \frac{1}{s(s^2 + a^2)}$$

$$L^{-1}[1/s(s^2+a^2)] = \int_0^t 1 \cdot \sin a(t-x)/a dx$$

$$= [-\cos(at-ax)/(-a^2)]_0^t$$

$$= (1/a^2)[1-\cos at]$$

Applications: Ordinary linear differential equations with constant coefficients can be easily solved by the Laplace Transform method, without finding the general solution and the arbitrary constants.

Solution of Differential Equation by Laplace Transform:

Problem: Using Laplace transforms, find the solution of the initial value problem

$$y'' - 4y' + 4y = 64\sin 2t, \quad y(0) = 0, y'(0) = 1$$

Solution: $y'' - 4y' + 4y = 64\sin 2t \ldots (1)$

Taking Laplace transform of both sides of (1), we have

$$[s^2\bar{y} - sy(0) - y'(0)] - 4[s\bar{y} - y(0)] + 4\bar{y} = (64 \times 2)/(s^2+4) \ldots (2)$$

On putting the value of $y(0)$ and $y'(0)$in (2), we get

$$s^2\bar{y} - 1 - 4s\bar{y} + 4\bar{y} = 128/(s^2+4)$$

$$(s^2 - 4s + 4)\bar{y} = 1 + 128/(s^2+4)$$

$$(s-2)^2\bar{y} = 1 + 128/(s^2+4)$$

$$\bar{y} = 1/(s-2)^2 + 128/[(s-2)^2(s^2+4)]$$

$$= 1/(s-2)^2 - 8/(s-2) + 16/(s-2)^2 + 8s/(s^2+4)$$

$$y = L^{-1}[-8/(s-2) + 17/(s-2)^2 + 8s/(s^2+4)]$$

$$y = -8e^{2t} + 17te^{2t} + 8\cos 2t$$

Problem: Using the Laplace transforms, find the solution of the initial value problem $y'' + 25y = 10\cos 5t, y(0) = 2, y'(0) = 0$

Solution. Taking Laplace transforms of the given differential eq., we get

$$[s^2\bar{y} - sy(0) - y'(0)] + 25\bar{y} = 10s/(s^2+25)$$

$$s^2\bar{y} - 2s + 25\bar{y} = 10s/(s^2+25)$$

$$(s^2+25)\bar{y} = 2s + 10s/(s^2+25)$$

$$\bar{y} = 2s/(s^2 + 25) + 10s/(s^2 + 25)^2$$

$$L^{-1}[\bar{y}] = L^{-1}[2s/(s^2 + 25) + 10s/(s^2 + 25)^2]$$

$$y = L^{-1}[2s/(s^2 + 25) + 10s/(s^2 + 25)^2]$$

$$= 2\cos 5t + L^{-1}[10s/(s^2 + 25)^2]$$

$$= 2\cos 5t + L^{-1}[(d/ds) - 5/(s^2 + 25)]$$

$$= 2\cos 5t + t\sin 5t$$

Problem: Applying convolution, solve the following initial value problem:

$$y'' + y = \sin 3t, y(0) = 0, y'(0) = 0$$

Solution. $y'' + y = \sin 3t$

Taking Laplace transformation of both the sides, we have

$$[s^2\bar{y} - sy(0) - y'(0)] + \bar{y} = 3/(s^2 + 9) \ldots (1)$$

On putting the values of y(0), y'(0) in (1) we get

$$s^2\bar{y} + \bar{y} = 3/(s^2 + 9)$$

$$(s^2 + 1)\bar{y} = 3/(s^2 + 9)$$

$$\bar{y} = 3/[(s^2 + 1)(s^2 + 9)]$$

$$= (3/8)[1/(s^2 + 1) - 1/(s^2 + 9)]$$

Taking the inversion transform we get

$$y = (3/8)L^{-1}[1/(s^2 + 1)] - (3/8)L^{-1}[1/(s^2 + 9)]$$

$$y = (3/8)\sin t - (3/8) \times (1/3)\sin 3t$$

$$= (3/8)\sin t - (1/8)\sin 3t$$

Problem: Using Laplace transforms, find the solution of the initial value problem

$$y'' + 9y = 6\cos 3t, y(0) = 2, y'(0) = 0$$

Solution. $y'' + 9y = 6\cos 3t \ldots (1)$

$$y(0) = 2, y'(0) = 0$$

Taking Laplace transform of (1), we get

$$[s^2\bar{y} - sy(0) - y'(0)] + 9\bar{y} = 6s/(s^2 + 9) \ldots (2)$$

Putting the value of $y(0)$ and $y'(0)$in (2), we have

$$s^2\bar{y} - 2s + 9\bar{y} = 6s/(s^2+9)$$

$$(s^2+9)\bar{y} = 2s + 6s/(s^2+9)$$

$$\text{or } \bar{y} = 2s/(s^2+9) + 6s/(s^2+9)^2$$

$$y = L^{-1}[2s/(s^2+9)] + L^{-1}[6s/(s^2+9)^2]$$

$$= 2\cos 3t + 3L^{-1}[(d/ds) - 3/(s^2+9)]$$

$$= 2\cos 3t - L^{-1}[(d/ds)3/(s^2+9)] = 2\cos 3t + t\sin 3t$$

Problem : Solve the initial value problem $2y'' + 5y' + 2y = e^{-2t}, y(0) = 1, y'(0) = 1$, using the Laplace transforms.

Solution. $2y'' + 5y' + 2y = e^{-2t}, y(0) = 1, y'(0) = 1$

Taking the Laplace Transform of both sides, we get

$$2[s^2\bar{y} - sy(0) - y'(0)] + 5[s\bar{y} - y(0)] + 2\bar{y} = 1/(s+2) \ldots (1)$$

On substituting the values $y(0)$ and $y'(0)$in (1), we get

$$2[s^2\bar{y} - s - 1] + 5[s\bar{y} - 1] + 2\bar{y} = 1/(s+2)$$

$$[2s^2 + 5s + 2]\bar{y} - 2s - 2 - 5 = 1/(s+2)$$

$$\bar{y} = 1/[(s+2)(2s^2+5s+2)] + (2s+7)/(2s^2+5s+2)$$

$$= (1 + 2s^2 + 7s + 4s + 14)/[(2s^2+5s+2)(s+2)]$$

$$= (2s^2 + 11s + 15)/[(2s+1)(s+2)^2]$$

$$= (4/9)/(2s+1) - (11/9)/(s+2)$$

$$- (1/3)/(s+2)^2$$

$$= (4/9)\cdot(1/2)/(s+1/2) - (11/9)/(s+2)$$

$$- (1/3)/(s+2)^2$$

$$y = (2/9)e^{-t} - (11/9)e^{-2t} - (1/3)te^{-2t}$$

Problem: Solve $d^2y/dx^2 + 2dy/dx + 5y = e^{-x}\sin x$ where $y(0) = 0, y'(0) = 1$.

Solution. $d^2y/dx^2 + 2dy/dx + 5y = e^{-x}\sin x$

Taking the Laplace Transforms of both the sides, we get

$$[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = L(e^{-x}\sin x)$$

$$[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = 1/[(s+1)^2 + 1] \ldots (1)$$

On substituting the values of $y(0)$ and $y'(0)$ in (1), we get

$$(s^2\bar{y} - 1) + 2(s\bar{y}) + 5\bar{y} = 1/(s^2 + 2s + 2)$$

$$(s^2 + 2s + 5)\bar{y} = 1 + 1/(s^2 + 2s + 2)$$

$$= (s^2 + 2s + 3)/(s^2 + 2s + 2)$$

$$\bar{y} = (s^2 + 2s + 3)/[(s^2 + 2s + 5)(s^2 + 2s + 2)]$$

On resolving the R.H.S. into partial fractions, we get

$$\bar{y} = (2/3)/(s^2 + 2s + 5) + (1/3)/(s^2 + 2s + 2)$$

On inversion, we obtain

$$y = (2/3)L^{-1}[1/(s^2 + 2s + 5)] + (1/3)L^{-1}[1/(s^2 + 2s + 2)]$$

$$y = (1/3)L^{-1}[2/(s+1)^2 + (2)^2] + (1/3)L^{-1}[1/(s+1)^2 + (1)^2]$$

$$\text{or } y = (1/3)e^{-x}\sin 2x + (1/3)e^{-x}\sin x \text{ or } y = (1/3)e^{-x}(\sin x + \sin 2x)$$

Problem: Using Laplace transforms, find the solution of the initial value problem $y'' + 9y = 9\,u(t-3), y(0) = y'(0) = 0$ where $u(t-3)$ is the unit step functions.

Solution. $y'' + 9y = 9\,u(t-3) \ldots (1)$

Taking Laplace transform of (1), we have

$$s^2\bar{y} - sy(0) - y'(0) + 9\bar{y} = 9e^{-3s}/s \ldots (2)$$

Putting the value of $y(0) = 0$ and $y'(0) = 0$ in (2), we get

$$s^2\bar{y} + 9\bar{y} = 9e^{-3s}/s$$

$$(s^2 + 9)\bar{y} = 9e^{-3s}/s$$

$$\bar{y} = 9e^{-3s}/[s(s^2 + 9)]$$

$$y = L^{-1}[9e^{-3s}/s(s^2 + 9)]$$

$$L^{-1}[3/(s^2 + 9)] = \sin 3t$$

$$3L^{-1}[3/s(s^2 + 9)] = 3\int_0^t \sin 3t\,dt$$

$$= -[\cos 3t]_0^t = 1 - \cos 3t$$

$$y = L^{-1}[9e^{-3s}/s(s^2 + 9)]$$

$$y = [1 - \cos 3(t-3)]u(t-3)$$

Solution of Simultaneous Differential Equations by Laplace Transforms:

Problem: Using Laplace transform, solve the simultaneous diff. eqn.

$$(D-2)x-(D+1)y=6e^{3t}$$
$$(2D-3)x-(D-3)y=6e^{3t}$$

given when t = 0, x = 3, & y = 0.

Solution: Taking Laplace transform of given equation, we get

$$LDx-2Lx-LDy-Ly=6Le^{3t}$$
$$2LDx-3Lx+LDy-3Ly=6Le^{3t}$$

$$\Rightarrow\begin{bmatrix} s\bar{x}-x(0)-2\bar{x}-s\bar{y}+y(0)-\bar{y}=6.\dfrac{1}{s-3} \\ 2s\bar{x}-2x(0)-3\bar{x}+s\bar{y}+y(0)-3\bar{y}=\dfrac{6}{s-3}\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix} (s-2)\bar{x}-(s+1)\bar{y}-3=\dfrac{6}{s-3} \\ (2s-3)\bar{x}+(s-3)\bar{y}-6=\dfrac{6}{s-3}\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix} (s-2)\bar{x}-(s+1)\bar{y}=\dfrac{3s-3}{s-3} \\ (2s-3)\bar{x}+(s-3)\bar{y}=\dfrac{6s-12}{s-3}\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix} (s-3)(s-2)\bar{x}-(s-3)(s+1)\bar{y}=3s-3 \\ (s+1)(2s-3)\bar{x}+(s+1)(s-3)\bar{y}=\dfrac{(s+1)(6s-12)}{s-3}\end{bmatrix}$$

On adding we get

$$(3s^2-6s+3)\bar{x}=3(s-1)+\frac{6(s^2-s-2)}{s-3}$$

$$\Rightarrow \bar{x}=\frac{3(s-1)}{3(s-1)^2}+\frac{6(s^2-s-2)}{3(s-1)^2(s-3)}$$

$$\Rightarrow x=L^{-1}\left[\frac{1}{(s-1)}+\frac{2}{(s-1)^2}+\frac{2}{(s-3)}\right]$$

$$=e^t+2te^t+2e^{3t}$$

Putting the value of x in (1) we get

$$(D-2)(e^t+2te^t+2e^{3t})-(D+1)y=6e^{3t}$$
$$\Rightarrow e^t+2te^t+2e^t+6e^{3t}-2e^t-4te^t-4e^{3t}-(D+1)y=6e^{3t}$$
$$\Rightarrow (D+1)y=e^t-2te^t-4e^{3t}$$

Taking Laplace transform, we get

$$s\bar{y}-y(0)+\bar{y}=\frac{1}{(s-1)}+\frac{2}{(s-1)^2}+\frac{4}{(s-3)}$$

$$\Rightarrow (s+1)\bar{y}=\frac{1}{(s-1)}+\frac{2}{(s-1)^2}+\frac{4}{(s-3)}$$

$$\Rightarrow \bar{y}=\frac{1}{s^2-1}-\frac{2}{(s+1)(s-1)^2}-\frac{4}{(s+1)(s-3)}$$

$$=\frac{1}{s^2-1}-\frac{\frac{1}{2}}{(s+1)}+\frac{\frac{1}{2}}{(s-1)}-\frac{1}{(s-1)}$$

$$\bar{y}=\frac{1}{s^2-1}+\frac{\frac{1}{2}}{(s+1)}+\frac{\frac{1}{2}}{(s-1)}-\frac{1}{(s-1)^2}+\frac{1}{(s+1)}-\frac{1}{(s-3)}$$

$$\bar{y}=\left[\frac{1}{s^2-1}+\frac{1}{2}\frac{1}{(s+1)}+\frac{1}{2}\frac{1}{(s-1)}-\frac{1}{(s-1)^2}-\frac{1}{(s-3)}\right]$$

$$y=\text{Sinh}\,t+\frac{1}{2}e^{-t}+\frac{1}{2}e^t-e^{3t}-te^t$$

so $y=\text{Sinh}\,t+\text{Cosh}\,t-e^{t3t}-te^t$

Problem: Solve $dx/dt+y=0$ and $dy/dt-x=0$under the condition $x(0)=1, y(0)=0$.

Solution. $x'+y=0\ldots(1)$

$$y'-x=0\ldots(2)$$

Taking the Laplace transform of (1) and (2) we get

$$[s\bar{x}-x(0)]+\bar{y}=0\ldots(3)$$

$$[s\bar{y}-y(0)]-\bar{x}=0\ldots(4)$$

On substituting the values of x(0) and y(0) in (3) and (4) we get

$$s\bar{x}-1+\bar{y}=0\ldots(5)$$

$$s\bar{y}-\bar{x}=0\ldots(6)$$

Solving (5) and (6) for $\bar{x}$ and $\bar{y}$ we get

$$\bar{x} = s/(s^2+1), \bar{y} = 1/(s^2+1)$$

On inversion, we obtain

$$x = L^{-1}[s/(s^2+1)], y = L^{-1}[1/(s^2+1)]$$

$$x = \cos t, y = \sin t$$

Problem: Using Laplace transforms, solve the differential equations

$$(D+1)y_1 + (D-1)y_2 = e^{-t}$$

$$(D+2)y_1 + (D+1)y_2 = e^{t}$$

where $D = d/dt$ and $y_1(0) = 1, y_2(0) = 0.$

Solution. $(D+1)y_1 + (D-1)y_2 = e^{-t} \ldots (1)$

$(D+2)y_1 + (D+1)y_2 = e^{t} \ldots (2)$

Multiply (1) by $(D+1)$ and (2) by $(D-1)$ we get

$(D+1)^2 y_1 + (D^2-1)y_2 = (D+1)e^{-t} \ldots (3)$

$(D-1)(D+2)y_1 + (D^2-1)y_2 = (D-1)e^{t} \ldots (4)$

Subtracting (4) from (3) we get

$(D^2 + 2D + 1 - D^2 - D + 2)y_1 = (-e^{-t} + e^{-t}) - (e^{t} - e^{t})$

or $(D+3)y_1 = 0$ or $Dy_1 + 3y_1 = 0$

Taking Laplace transforms we have

$s\bar{y}_1 - y_1(0) + 3\bar{y}_1 = 0$

$(s+3)\bar{y}_1 = 1$ or $\bar{y}_1 = 1/(s+3)$

$y_1 = e^{-3t}$

Laplace Transformation

$$y = -\tfrac{1}{2}L^{-1}[1/(s-1)] + \tfrac{1}{2}L^{-1}[s/(s^2+1)] - \tfrac{1}{2}L^{-1}[1/(s^2+1)]$$
$$+ L^{-1}[s/(s^2+1)^2] y = -\tfrac{1}{2}e^{t} + \tfrac{1}{2}\cos t - \tfrac{1}{2}\sin t + \tfrac{1}{2}t\sin t$$

$$y = \tfrac{1}{2}[-e^{t} - \sin t + \cos t + t\sin t]$$

Problem: Using the Laplace transforms, solve the initial value problem

$$y_1'' = y_1 + 3y_2$$

$$y_2'' = 4y_1 - 4e^{t}$$

$$y_1(0) = 2, y_1'(0) = 3, y_2(0) = 1, y_2'(0) = 2.$$

Solution: $$y_1'' = y_1 + 3y_2 \ldots (1)$$

$$y_2'' = 4y_1 - 4e^t \ldots (2)$$

Taking the Laplace transform of (1) and (2):

$$s^2\bar{y}_1 - sy_1(0) - y_1'(0) = \bar{y}_1 + 3\bar{y}_2 \ldots (3)$$

$$s^2\bar{y}_2 - sy_2(0) - y_2'(0) = 4\bar{y}_1 - 4/(s-1) \ldots (4)$$

Putting the value of $y_1(0), y_1'(0), y_2(0), y_2'(0)$ in (3) and (4), we get:

$$s^2\bar{y}_1 - 2s - 3 = \bar{y}_1 + 3\bar{y}_2$$

$$\text{or} \quad (s^2 - 1)\bar{y}_1 - 3\bar{y}_2 = 2s + 3 \ldots (5)$$

$$s^2\bar{y}_2 - s - 2 = 4\bar{y}_1 - 4/(s-1)$$

$$\text{or} \quad 4\bar{y}_1 - s\bar{y}_2 = 4/(s-1) - s - 2 \ldots (6)$$

On solving (5) and (6), we get:

$$\bar{y}_1 = [(2s-3)(s^2+3)(s+2)]/[(s-1)(s^2+3)(s^2-4)]$$

$$= (2s-3)/[(s-1)(s-2)] = 1/(s-1) + 1/(s-2)$$

$$y_1 = e^t + e^{2t}$$

$$\bar{y}_2 = [(s+2)(s^2+3)]/[(s^2+3)(s^2-4)]$$

$$= 1/(s-2)$$

$$\Rightarrow y_2 = e^{2t}$$

Unit–III

Tutorial Sheet

Fourier Series

1. Find a Fourier series to represent $f(x) = \pi - x$ for $0 < x < 2\pi$.
   Ans. $2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots + \frac{1}{n}\sin nx + \cdots\right]$
2. Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to π and show that:
   $$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$
   Ans. $-\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$
3. Find a Fourier series to represent: $f(x) = x\sin x$, for $0 < x < 2\pi$.
   Ans. $-1 + \pi\sin x - \frac{1}{2}\cos x + 2\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} + \cdots\right]$
4. Find a Fourier series to represent the function $f(x) = e^x$, for $-\pi < x < \pi$ and hence derive a series for $\frac{\pi}{\sinh\pi}$.
   Ans. $\frac{2\sinh\pi}{\pi}\left[\left(\frac{1}{2} - \frac{1}{1^2+1}\cos x + \frac{1}{2^2+1}\cos 2x - \frac{1}{3^2+1}\cos 3x + \cdots\right) + \frac{1}{1^2+1}\sin x - \frac{2}{2^2+1}\sin 2x + \frac{3}{3^2+1}\sin 3x \ldots\right]$; $\frac{\pi}{\sinh\pi} = 1 + 2\left[-\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \cdots\right]$
5. Find the Fourier series of the function:
   $$f(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi \end{cases} \quad \text{where } f(x + 2\pi) = f(x).$$
   Ans. $\frac{4}{\pi}\left[\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots\right]$
6. Find the Fourier series for the function: $f(x) = \begin{cases} -\frac{\pi}{4} & \text{for } -\pi < x < 0 \\ \frac{\pi}{4} & \text{for } 0 < x < \pi \end{cases}$
   and $f(-\pi) = f(0) = f(\pi) = 0, f(x) = f(x + 2\pi)$ for all x.
   Deduce that:
   $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ **Ans**. $\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots$
7. Find the Fourier series of the function: $f(x) = \begin{cases} 0 & \text{for } -\pi \le x \le 0 \\ 1 & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} \le x \le \pi \end{cases}$
8. Obtain a Fourier series to represent the following periodic function:

$$f(x) = 0 \quad \text{when } 0 < x < \pi$$
$$f(x) = 1 \quad \text{when } \pi < x < 2\pi$$

Ans. $\frac{1}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$

9. Find the Fourier expansion of the function defined in a single period by the relations: $f(x) = \begin{cases} 1 & \text{for } 0 < x < \pi \\ 2 & \text{for } \pi < x < 2\pi \end{cases}$

   and from it deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

   Ans. $\frac{3}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$

10. Find a Fourier series to represent the function

$$f(x) = \begin{cases} 0 & \text{for } -\pi < x \le 0 \\ \frac{1}{4}\pi x & \text{for } 0 < x < \pi \end{cases}$$

   and hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

   Ans. $\frac{\pi^2}{16} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{4n^2}\cos nx - \frac{(-1)^n \pi}{4n}\sin nx + \cdots\right]$

11. Find the Fourier series for f(x), if: $f(x) = -\pi \quad \text{for } -\pi < x \le 0$
$$= x \quad \text{for } 0 < x < \pi$$
$$= -\frac{\pi}{2} \quad \text{for } x = 0$$

   Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

   Ans. $-\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + 3\sin x - \frac{1}{2}\sin 2x + \frac{3}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots$

12. Obtain a Fourier series to represent the function: $f(x) = |x| \quad \text{for } -\pi < x < \pi$

   and hence deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

   Ans. $\frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$

13. Expand as a Fourier series, the function f(x)defined as

$$f(x) = \begin{cases} \pi + x & \text{for } -\pi < x < -\frac{\pi}{2} \\ \frac{\pi}{2} & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$$

Ans. $\frac{3\pi}{8} + \frac{2}{\pi}\left[\frac{1}{1^2}\cos x - \frac{2}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right]$

14. Obtain a Fourier series to represent the function

$$f(x) = |\sin x| \quad \text{for } -\pi < x < \pi$$

$$\left[\text{Hint } f(x) = \begin{cases} -\sin x & \text{for } -\pi < x < 0 \\ \sin x & \text{for } 0 < x < \pi \end{cases}\right]$$

$$\text{Ans. } \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x + \cdots\right]$$

15. An alternating current after passing through a rectifier has the form

$$i = I\sin\theta \quad \text{for } 0 < \theta < \pi$$

$$= 0 \quad \text{for } \pi < \theta < 2\pi$$

16. Find the Fourier series of the function.

$$\text{Ans. } \frac{I}{\pi} - \frac{2I}{\pi}\left(\frac{\cos 2\theta}{3} + \frac{\cos 4\theta}{15} + \cdots\right) + \frac{I}{2}\sin\theta$$

17. If $f(x) = 0$ for $-\pi < x < 0$ and $= \sin x$ for $0 < x < \pi$

Prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2-1}$.

Hence show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots\infty = \frac{1}{4}(\pi - 2)$

18. Find the Fourier cosine series for the function

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \dfrac{\pi}{2} \\ 0 & \text{for } \dfrac{\pi}{2} < x < \pi \end{cases}$$

$$\text{Ans. } \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \cdots\right]$$

19. Find a series of cosine of multiples of x which will represent $f(x)$ in $(0, \pi)$ where $f(x) = 0 \quad \text{for } 0 < x < \frac{\pi}{2}$

$$f(x) = \frac{\pi}{2} \quad \text{for } \frac{\pi}{2} < x < \pi$$

$$\text{Deduce that } 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\infty = \frac{\pi}{4}$$

$$\text{Ans. } \frac{\pi}{4} - \cos x + \frac{1}{3}\cos 3x - \frac{1}{5}\cos 5x + \cdots$$

20. Express $f(x) = x$ as a sin series in $0 < x < \pi$.

$$\text{Ans. } 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$$

21. Find the cosine series for $f(x) = \pi - x$ in the interval $0 < x < \pi$.

Ans. $\frac{\pi}{2}+\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right)$

22. If $f(x)=x, \quad$ for $0<x<\frac{\pi}{2}$

$$=\pi-x, \quad \text{for } \frac{\pi}{2}<x<\pi$$

Show that:(i) $f(x)=\frac{4}{\pi}\left(\sin x-\frac{1}{3^2}\sin 3x+\frac{1}{5^2}\sin 5x-\cdots\right)$

(ii) $f(x)=\frac{\pi}{4}-\frac{2}{\pi}\left(\frac{1}{1^2}\cos 2x+\frac{1}{3^2}\cos 6x+\frac{1}{5^2}\cos 10x+\cdots\right)$

23. Obtain the half-range cosine series for $f(x)=x^2$ in $0<x<\pi$.

Ans. $\frac{\pi^2}{3}-\frac{4}{\pi}\left(\cos x-\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x-\cdots\right)$

24. Find (i) sine series and (ii) cosine series for the function

$$f(x)=e^x \quad \text{for } 0<x<\pi.$$

Ans. (i) $\frac{2}{\pi}\sum_{n=1}^{\infty} n\left[\frac{1-(-1)^n e^{\pi}}{n^2+1}\right]\sin nx$ (ii) $\frac{e^{\pi}-1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n e^{\pi}}{n^2+1}\cos nx$

25. If $f(x)=x+1$, for $0<x<\pi$, find its Fourier (i) sine series (ii) cosine series. Hence deduce that

(i) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$ (ii) $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}$

Ans. (i) $\frac{2}{\pi}\left[(\pi+2)\sin x-\frac{\pi}{2}\sin 2x+\frac{1}{3}(\pi+2)\sin 3x-\frac{\pi}{4}\sin 4x+\cdots\right]$

(ii) $\frac{\pi}{2}+1-4\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right)$

26. Find the Fourier series expansion of the function

$$f(x)=\cos(sx), \quad -\pi\le x\le\pi$$

where s is a fraction. Hence, show that $\cot\theta=\frac{1}{\theta}+\frac{2\theta}{\theta^2-\pi^2}+\frac{2\theta}{\theta^2-4\pi^2}+\cdots$

Ans. $\frac{\sin\pi s}{\pi s}+\frac{1}{\pi}\left(\frac{\sin(s\pi+n\pi)}{s+n}+\frac{\sin(s\pi-n\pi)}{s-n}\right)\cos nx$

27. Find the Fourier series to represent f(x), where

$$f(x)=-a \quad -c<x<0$$

$$=a \quad 0<x<c$$

Ans. $\frac{4a}{\pi}\left[\sin\frac{\pi x}{c}+\frac{1}{3}\sin\frac{3\pi x}{c}+\frac{1}{5}\sin\frac{5\pi x}{c}+\cdots\right]$

28. Express $f(x) = x$ as a cosine, half range series in $0 < x < 2$.

$$\text{Ans. } 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$$

29. Find the Fourier series of the function

$$f(x) = \begin{cases} -2 & \text{for } -4 < x < -2 \\ x & \text{for } -2 < x < 2 \\ 2 & \text{for } 2 < x < 4 \end{cases}$$

$$\text{Ans. } \frac{4}{\pi} + \frac{8}{\pi^2}\sin\frac{\pi x}{4} - \frac{2}{\pi}\sin\frac{2\pi x}{4} + \left(\frac{4}{3\pi} - \frac{8}{3^2\pi}\right)\sin\frac{3\pi x}{4} - \frac{2}{2\pi}\sin\frac{4\pi x}{4} + \cdots$$

30. Find the Fourier series to represent

$$f(x) = x^2 - 2 \quad \text{from } -2 < x < 2.$$

$$\text{Ans. } -\frac{2}{3} - \frac{16}{\pi^2}\left[\cos\frac{\pi x}{2^2} - \frac{1}{4}\cos\pi x + \frac{1}{9}\cos\frac{3\pi x}{2} \ldots\right]$$

Laplace Transforms

Find Laplace transform of:

1. $t \sin at$ — Ans. $\frac{2as}{(s^2 + a^2)^2}$.

2. $\left(\frac{1}{t}\right)(1 - e^t)$ — Ans. $\log\left[\frac{s - 1}{s}\right]$

3. $\left(\frac{1}{t}\right)\left(e^{-at} - e^{-bt}\right)$ — Ans. $\log\left[\frac{s + b}{s + a}\right]$

4. $f(t) = t - 1 (1 < t < 2)$, 0 otherwise — Ans $\left(e^{(-s)} - e^{(-2s)}\right)/s^2 - \left(e^{(-2s)}/s\right)$

5. $e^t\, u(t - 1)$ — Ans. $e^{(-(s-1))}/(s - 1)$

6. $t^2\, u(t - 2)$ — Ans. $\left(e^{(-2s)}/s^3\right)(4s^2 + 4s + 2)$

7. $\sin t\, u(t - 4)$ — Ans. $\left(e^{(-4s)}/(s^2 + 1)\right)[\cos 4 + s\sin 4]$

8. $f(t) = K(t - 2)[U(t - 2) - U(t - 3)]$ — Ans. $(K/s^2)\left[e^{(-2s)} - (s + 1)e^{(-3s)}\right]$

9. $\int_0^\infty e^{-3t}\,\delta(t - 4)dt$ — Ans. e^{-12}.

10. $\int_0^\infty \sin 2t\,\delta(t - \pi/4)dt$ — Ans. 1

11. $\int_{-\infty}^\infty e^{-3t}\,\delta'(t - 2)dt$ — Ans. $3e^{(-6)}$

12. $\delta(t-4)/t$ Ans. $e^{(-4s)}/4$

13. $e^{(-4t)}\delta(t-3)$ Ans. $e^{(-3(s+4))}$

14. $f(t)=e^{t}$ for $0<t<2\pi$. Ans. $\left(e^{2(1-s)\pi}-1\right)/[(1-s)(1-e^{-2\pi s})]$

15. Full wave rectified sin wave: $f(t)=\sin\omega t$ $(0<t<\pi/\omega)$

Ans. $\left(\omega/(s^2+\omega^2)\right)\coth(\pi s/2\omega)$

16. Staircase function: $f(t)=kn,\ np<t<(n+1)p$ Ans. $ke^{(ps)}/[s(1-e^{-ps})]$

17.$f(t)=t^2,\ 0<t<2,\quad f(t+2)=f(t)$

Ans. $(2-e^{-2s}-4se^{-2s}-4s^2e^{-2s})/[s^3(1-e^{-2s})]$

18. $f(t)=\begin{cases} t\ , 0<t<c \\ 2c-t,\ c<t<2c \end{cases}$ Ans. $(1/s^2)\tanh(cs/2)$

19.$f(t)=\begin{cases} \cos\omega t, & 0<t<\pi/\omega \\ 0\ , & \pi/\omega<t<2\pi/\omega \end{cases}$

Ans. $\left(s/(s^2+\omega^2)\right)\left(1-e^{-\pi s/\omega}\right)$

Find the inverse Laplace transform of:

1. $s/(s+5)$ Ans. $-5e^{(-5t)}+1$

2. $2s/(3s+6)$ Ans. $(2/3)\left[-2e^{(-2t)}+1\right]$

3.$s/(2s^2-1)$ Ans. $(1/2)\cosh\left(t/\sqrt{2}\right)$

4. $s^2/(s^2+a^2)$ Ans. $-a\sin at+1$

5. $(s^2+4)/(s^2+9)$ Ans. $-(5/3)\sin3t+1$

6. $[s^2/(s^2+4)^2]$ Ans. $(1/4)\sin2t+(t/2)\cos2t$

7. $1/[2s(s-3)]$ Ans. $(1/2)\left[e^{(3t)}/3-1\right]$

8. $1/[s(s+2)]$ Ans. $\left(1-e^{(-2t)}\right)/2$

9. $1/[s(s^2-16)]$ Ans. $(1/16)[\cosh4t-1]$

10. $1/[s(s^2+a^2)]$ Ans. $(1-\cos at)/a^2$

11. $(s^2+2)/[s(s^2+4)]$ Ans. $\cos^2t$

12. $(s+8)/(s^2+4s+5)$ Ans. $e^{(-2t)}(\cos t+6\sin t)$

13. $s/[(s+3)^2+4]$ Ans. $e^{(-3t)}(\cos2t-1.5\sin2t)$

14. $s/(s+7)^4$ Ans. $e^{(-7t)}(t^2/6)(3-7t)$

15. $(s+2)/(s^2-2s-8)$ Ans. $e^{t}(\cosh3t+\sinh3t)$

16. $s/(s^2+6s+25)$ Ans. $e^{(-3t)}[\cos 4t-(3/4)\sin 4t]$

17. $1/[2(s-1)^2+32]$ Ans. $(e^t/8)\sin 4t$

18. $(s-4)/[4(s-3)^2+16]$ Ans. $(1/4)e^{(3t)}\cos 2t-(1/8)e^{(3t)}\sin 2t$

19. $\left[se^{(-s/2)}+\pi e^{(-s)}\right]/(s^2+\pi^2)$ Ans. $[U(t-1/2)-U(t-1)]\sin \pi t$

20. $e^{(-s)}/(s+2)^3$ Ans. $e^{(-(t-2))}[(t-2)^2/2]U(t-2)$

21. $e^{(-2s)}/[(s+1)(s^2+2s+2)]$ Ans. $e^{(-(t-2))}[1-\cos(t-2)]U(t-2)$

22. $e^{(-s)}/\sqrt{(s+1)}$ Ans. $e^{(-(t-1))}/\sqrt{(\pi(t-1))}U(t-1)$

23. $\left[e^{(-\pi s/2)}+e^{(-3\pi s/2)}\right]/(s^2+1)$ Ans. $\cot[U(t-3\pi/2)-U(t-\pi/2)]$

24. $e^{(-4s)}(s+2)/(s^2+4s+5)$ Ans. $e^{(-2(t-4))}\cos(t-4)U(t-4)$

25. $e^{(-as)}/s^2$ Ans. $f(t)=t-a$ when $t>a$; 0 when $t<a$

26. $e^{(-\pi s)}/(s^2+1)$ Ans. $-\sin tu(t-\pi)$

27. $\log(1+\omega^2/s^2)$ Ans. $-(2/t)\cos\omega t+2$

28. $\log(1+1/s^2)$ Ans. $(2/t)[1-\cos t]$

29. $s/(1+s^2+s^4)$ Ans. $\left(2/\sqrt{3}\right)\sin\left(\sqrt{3}t/2\right)\sinh(t/2)$

30. $(s+1)/(s^2+6s+13)^2$ Ans. $\left(e^{(-3t)}/8\right)[2t\sin 2t+2t\cos 2t-\sin 2t]$

31. $s/(s^2+a^2)^2$ Ans. $(t\sin at)/2a$

32. $s\log\left[s/\sqrt{(s^2+1)}\right]+\cot^{-1}s$ Ans. $(1-\cos t)/t^2$

33. $(1/2)\log[(s^2+b^2)/(s-a)^2]$ Ans. $\left(e^{(at)}-\cos bt\right)/t$

34. $\tan^{-1}(s+1)$ Ans. $-(1/t)e^{(-t)}\sin t$

35. $(s^2+2s+6)/s^3$ Ans. $1+2t+3t^2$

36. $1/(s^2-7s+12)$ Ans. $e^{4t}-e^{3t}$

37. $(s+2)/(s^2-4s+13)$ Ans. $e^{2t}\cos 3t+(4/3)e^{2t}\sin 3t$

38. $(3s+1)/[(s-1)(s^2+1)]$ Ans. $e^t-2\cos t+\sin t$

39. $(11s^2-2s+5)/(2s^3-3s^2-3s+2)$ Ans. $2e^{-t}+5e^{2t}-(3/2)e^t/^2$

40. $(2s^2-6s+5)/[(s-1)(s-2)(s-3)]$ Ans. $(1/2)e^t-e^{2t}+(5/2)e^{3t}$

41. $(s-4)/[(s-4)^2+9]$ Ans. $e^{4t}\cos 3t$

42. $16/(s^2+2s+5)^2$ Ans. $e^{-t}(\sin 2t-2t\cos 2t)$

43. $1/[(s+1)(s^2+2s+2)]$ Ans. $e^{-t}(1-\cos t)$

44. $1/[(s-2)(s^2+1)]$ Ans. $(1/5)e^{2t}-(1/5)\cos t-(2/5)\sin t$

46. $s^2/(s^2+a^2)^2$ Ans. $(1/2)t\cos at+(1/2a)\sin at$

47. $1/(s^2+1)^3$ Ans. $(1/8)[(3-t^2)\sin t-3t\cos t]$

Solving the following differential equations using Laplace transform:

1. $d^2y/dx^2 + y = 0, \text{where} y = 1 \text{and} dy/dx = -1 \quad \text{at} \quad x = 0.$

$$\text{Ans.} y = \cos x - \sin x$$

2. $d^2y/dx^2 - 4y = 0, \text{where} \ y = 0 \text{and} d \ \ y/dx = -6 \quad \text{at} \quad x = 0.$

$$\text{Ans.} y = -(3/2)e^{2x} + (3/2)e^{-2x}$$

3. $d^2y/dx^2 + y = 0, \text{where} y = 1, dy/dx = 1 \text{at} x = 0.$

$$\text{Ans.} y = \sin x + \cos x$$

4. $d^2y/dx^2 + 2dy/dx + 5y = 0, \text{where} y = 2, dy/dx = -4 \text{at} x = 0.$

$$\text{Ans.} y = e^{-x}(2\cos 2x - \sin 2x)$$

5. $d^3y/dx^3 + 2d^2y/dx^2 - dy/dx - 2y = 0, \text{given} \ y = 0, \ dy/dx = 0, d^2y/dx^2 = 6 \text{ at} \quad x = 0.$ $\text{Ans.} y = e^x - 3e^{-x} + 2e^{-2x}$

6. $d^2y/dx^2 + y = 3\cos 2x, \text{where} y = dy/dx = 0 \text{at} x = 0.$

$$\text{Ans.} y = \cos x - \cos 2x$$

7. $d^2y/dx^2 + dy/dx - 2y = 1 - 2x, \text{given} y = 0, dy/dx = 4 \text{at} x = 0.$

$$\text{Ans.} y = e^x - e^{-2x} + x$$

8. $d^2y/dx^2 - 3dy/dx + 2y = 4e^{2x}, \text{given} y = -3, \text{and} dy/dx = 5 \text{at} x = 0.$

$$\text{Ans.} y = -7e^x + 4e^{2x} + 4xe^{2x}$$

9. $d^2y/dx^2 - 3dy/dx + 2y = 4x + e^{2x}, \text{where} y = 1, dy/dx = -1 \text{at} x = 0.$

$$\text{Ans.} y = 3 + 2x + (1/2)e^{3x} - 2e^{2x} - (1/2)e^x$$

10. $d^3y/dx^3 + 2d^2y/dx^2 - dy/dx - 2y = 0, \text{where} \ y = 1, dy/dx = 2, \ d^2y/dx^2 = 2 \text{ at } x = 0.$ $\text{Ans.} (5/3)e^x - e^{-x} + (1/3)e^{-2x}$

11. $(D^2 - D - 2)x = 20\sin 2t, \ x_0 = -1, \ x_1 = 2.$

$$\text{Ans.} x = 2e^{2t} - 4e^{-t} + \cos 2t - 3\sin 2t$$

12. $(D^3 + D^2)x = 6t^2 + 4, x(0) = 0, x'(0) = 2, x''(0) = 0$

$$\text{Ans.} x = (1/2)t^4 - 2t^3 + 8t^2 - 16t + 16 - 16e^{-t}$$

13. $d^2x/dt^2 - 2dx/dt + x = e^t, \text{where} \ x(0) = 2, \ dx/dt = -1 \ \text{at} \ t = 0.$

$$\text{Ans.} x = 2e^t - 3te^t + (1/2)t^2e^t$$

14. $(D^2 + n^2)x = a\sin(nt + \alpha) \text{where} \ x = Dx = 0 \ \text{at} \ t = 0.$

$$\text{Ans.} x = a_n \cos\alpha(\sin nt - nt\cos nt) + (a\sin 2\alpha/2n)(t\sin nt)$$

15. $y'' + 2y' + y = te^{-t} \quad \text{if} \ y(0) = 1, y'(0) = -2. \quad \text{Ans.} y = (1 - t + t^3/6)e^{-t}$

16. $d^2y/dx^2 + y = xcos2x, where \ \ y = dy/dx = 0 \ atx = 0.$

$$\text{Ans.} \quad y = (4/9)\sin 2x - (5/9)\sin x - (1/3)x\cos 2x$$

17. $d^3y/dx^3 - 3d^2y/dx^2 + 3dy/dx - y = x^2e^{2x}, \text{where} \ y = 1, dy/dx = 0, d^2y/dx^2 = -2 \quad \text{at} \ x = 0.$ $\text{Ans.} y = e^{2x}(x^2 - 6x + 12) - e^x(15x^2 + 7x + 11)$

18. $y'' + 4y' + 3y = t, t > 0; \text{given that} \ y(0) = 0 \ \text{and} \ y'(0) = 1.$

$$\text{Ans.} y = -4/9 + t/6 + e^{-t} - (5/9)e^{-3t}$$

19. $y'' + 2y = r(t), y(0) = 0, y'(0) = 0$ where $r(t) = 0$, $t \geq 1; 1,0 \leq t < 1.$

$$\text{Ans. } y = 1/2 - (1/2)\cos\sqrt{2}t$$

20. $d^2y/dx^2 + y = u(t - \pi) - u(t - 2\pi), y(0) = y'(0) = 0.$

$$\text{Ans. } y = (1 + \cos t)u(t - \pi) - (1 - \cos t)u(t - 2\pi)$$

Solving the following simultaneous differential equations using Laplace transform:

1. $dx/dt + 4y = 0, dy/dt - 9x = 0.$ Given $x = 2$ and $y = 1$ at $t = 0.$

$$\text{Ans. } x = -\frac{2}{3}\sin 6t + 2\cos 6t, \qquad y = \cos 6t + 3\sin 6t$$

2. $4(dy/dt) + (dx/dt) + 3y = 0,\ 3(dx/dt) + 2x + (dy/dt) = 1$ under the condition $x = y = 0$ at $t = 0.$

$$\text{Ans. } x = \frac{1}{2} - \frac{1}{5}e^{-t} - \frac{3}{10}e^{-(6/11)t}\ , \qquad y = \frac{1}{5}e^{-t} - \frac{1}{5}e^{-(6/11)t}$$

3. $dx/dt + 5x - 2y = t, dy/dt + 2x + y = 0$ being given $x = y = 0$ when $t = 0.$

$$\text{Ans. } x = \frac{-1}{27}(1 + 6t)e^{-3t} + \frac{1}{27}(1 + 3t), \qquad y = \frac{-2}{27}(2 + 3t)e^{-3t} - \frac{2t}{9} + \frac{4}{27}$$

4. $dx/dt + y = \sin t, dy/dt + x = \cos t$

given that $x = 2,$ and $y = 0$ when $t = 0.$ Ans. $x = e^t + e^{-t}, y = e^{-t} - e^t + \sin t$

5. The small oscillations of a certain system with two degrees of freedom are given by the equations:

$$D^2x + 3x - 2y = 0, \qquad D^2x + D^2y - 3x + 5y = 0\ , \text{if } x = 0, y = 0, Dx = 3,$$

$$Dy = 2 \quad \text{when} \quad t = 0.$$

$$\text{Ans. } x = \frac{-11}{4}\sin t + \frac{1}{12}\sin 3t, \quad y = \frac{11}{4}\sin t - \frac{1}{4}\sin 3t.$$

6. $3(dx/dt) + 3(dy/dt) + 5x = 25\cos t,$

$2(dx/dt) - 3(dy/dt) = 5\sin t$ with $x(0) = 2, y(0) = 3.$

$$\text{Ans. } x = 2\cos t + 3\sin t, \qquad y = 3\cos t + 2\sin 2t.$$